Chapter 22: Epistemologies of Mathematics and of Mathematics Education

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ABSTRACT

This chapter addresses issues concerning epistemology, as they relate to mathematics and education. It commences with an examination of some of the main epistemological questions concerning truth, meaning and certainty, and the different ways they can be interpreted. It examines epistemologies of the 'context of justification' and of the 'context of discovery', foundationalist and non-foundationalist epistemologies of mathematics, historico-critical, genetic, socio-historical and cultural epistemologies, and epistemologies of meaning.

In the second part of the chapter, after a brief look at epistemology in relation to the statements *of* mathematics education, epistemologies *in* mathematics education become the main focus of attention. Controversial issues within a number of areas are considered: the subjective-objective character of mathematical knowledge; the role in cognition of social and cultural context; and relations between language and knowledge. The major tenets of constructivism, socio-cultural views, interactionism, the French *didactique*, and epistemologies of meaning are compared. Relationships between epistemology and a theory of instruction, especially in regard to didactic principles, are also considered.

1. EPISTEMOLOGIES OF MATHEMATICS AND OF MATHEMATICS EDUCATION

Our intention in this chapter is to clarify what is meant by 'epistemology' in the various settings in which it is used within the international mathematics education community, to elaborate critically the origins, meanings and uses of those notions of epistemology, and to reflect on our practices as researchers and educators in relation to the epistemological theories upon which we draw. We shall not attempt to be exhaustive in our study of epistemologies of mathematics and mathematics education, neither in a historical sense nor in our examination of current theories.

The chapter has been written primarily for mathematics educators – not for philosophers of mathematics – and we shall confine our study to those areas

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that we consider relevant to our audience (and ourselves). Although our review of the research done in mathematics education in relation to epistemology will not be exhaustive, nonetheless major epistemological issues are addressed.

The chapter commences with an overview of the basic questions of epistemology and the many different ways in which they can be interpreted. In fact, the first part of the chapter ('Epistemologies of Mathematics') attempts to 'sort out' epistemologies. In particular, we evoke historical discussions related to the distinction between epistemology on the one hand, and psychology, sociology and history of science, on the other. This leads us to speak about epistemologies of the 'context of justification' and epistemologies of the 'context of discovery'. Reference is made to foundationalist and non-foundationalist epistemologies of mathematics, as well as to historico-critical, genetic, socio-historical and cultural epistemologies, and to epistemologies of meaning.

In the second part of the chapter ('Epistemologies of Mathematics Education') we proceed to an examination of the use and role of epistemologies in mathematics education. We argue that constructivist, socio-cultural, interactionist and anthropological approaches are founded on different epistemologies of knowledge. We also discuss approaches that focus on epistemological analyses of the meaning of particular mathematical concepts. We end with a reflection on relations between epistemology and theories of instruction which necessarily incorporate systems of values or principles. This discussion will include the issues of complementarity and eclecticism.

1.1 Epistemologies of Mathematics

In this chapter we shall be concerned with the clarification of the notion of epistemology itself, its various meanings, questions considered epistemological and not, and different interpretations of these questions.

1.1.1 Sorting Out Epistemological Questions

Epistemology as a branch of philosophy concerned with scientific knowledge poses fundamental questions such as: 'What are the origins of scientific knowledge?' (Empirical? Rational?); 'What are the criteria of validity of scientific knowledge?' (Able to predict actual events? Logical consistency?); 'What is the character of the process of development of scientific knowledge?' (Accumulation and continuity? Periods of normal science, scientific revolutions and discontinuity? Shifts and refinement in scientific programs?).

These questions can be interpreted in various ways. They can be asked in their full generality, as above, or they can be made more specific with respect to some particular domain of scientific knowledge, for example, mathematics. One can also be interested in knowledge from various perspectives. One can ask: what are the origins of the validity of our beliefs? Or, what are the sources of meaning of knowledge, and how is the meaning constituted? These are different questions because meaning and truth are different categories. One can also ask: what is the ontogenesis of knowledge? and speak of the development of 'cognitive structures', for example. Or the question can be posed about the 'phylogenesis' of discursive systems of knowledge such as mathematics or its parts.

Some prefer to approach epistemological questions in a philosophical way, and others in a more scientific way. The former ask: How can a scientific result be rationally explained on the basis of what it was obtained from? The latter ask: How was a given scientific result actually obtained?

These questions discriminate between the attitudes towards epistemology of mathematical foundationalists and mathematics educators. Mathematics educators are generally less interested in studying grounds for the validity of mathematical theories than in explaining the processes of growth of mathematical knowledge: their mechanisms, the conditions and contexts of past discoveries, the causes of periods of stagnation and claims that, from the point of view of present day theory, appear to be, or have been, erroneous.

Mathematics educators are also interested in observing and explaining the processes of mathematical discovery in the making, both in expert mathematicians and in students. Ultimately, as practitioners, they research ways of provoking such processes in teaching. If questions of certainty occupy mathematics educators, it is often in the context of discussing the concept of error, its different categories and the possible undertakings of the teacher in reaction to students' errors, misconceptions or conceptions departing from those accepted or expected. However, as will be shown, there have been a number of studies of the significance of philosophical issues for mathematics education.

All mathematics educators do not share the same epistemology, even if they are concerned with similar epistemological questions. We shall see in the second part of this chapter that the lines of division lie along issues such as the subjective-objective character of knowledge, the role in cognition of the social and cultural context, and the relations between language and knowledge.

1.1.2 Epistemology of the Context of Justification and Foundationalism in the Philosophy of Mathematics

The above concerns of mathematics educators would have been regarded, by certain philosophers of science from the first half of the century, as not belonging to epistemology proper but to psychology, history, sociology or semiotics. For example, Carnap (1928/1966), and Reichenbach (1938/1947) proposed that epistemology occupies itself with a 'rational reconstruction' of scientific thought processes, that is to say with the description of how scientific processes would develop if 'irrational factors' did not interfere. 'Rational reconstructions' were meant to be descriptions of the thinking processes of scientists not when they are discovering something, but when they are trying to communicate and justify their findings. They were meant to be accounts of the 'context of justification' of scientific thought. The 'context of discovery' or the actual processes of scientific discovery and the impact on them of cognitive, social and cultural-historical factors belongs, according to these authors, not to epistemology but to the empirical domains of psychology, sociology and history of knowledge.

Karl Popper (1972) understood epistemology in an 'anti-psychologistic' way. Imre Lakatos, a disciple and critic of Popper, extended the domain of epistemological reconstruction to those parts of the discovery process that he felt could be rationalised. His *Proofs and Refutations* (1976) provided a rational reconstruction of processes of discovery and justification of a certain part of topology. But Lakatos' epistemology remains programmatically anti-psychologistic. His notion, for example, of the 'proof-generated concept' is a methodological tool in rational reconstructions, not a generalisation of historical or psychological facts.

A 'context of justification' approach to epistemology in the philosophy of mathematics is known as 'foundationalism'. The foundationalist approach to the questions of growth of mathematics is a-historical and a-social: 'the history of mathematics is punctuated by events in which individuals are illuminated by the new insights that bear no particular relation to the antecedent of the discipline' (Kitcher, 1988).

Answers to the question of origins of knowledge have been traditionally put into two categories: apriorism and empiricism. Foundationalist philosophies of mathematics whose main concern is to find some 'first mathematics, some special discipline from which all the rest must be built' (Kitcher, 1988, p.294) are necessarily aprioristic. Otherwise, says Kitcher, there would be no point in their concerns. Of course, apriorism appears as a sensible solution, if empiricism, especially a naive empiricism, is seen as the only epistemological alternative. Empiricism is simply unacceptable for an epistemology of mathematics. Richard (1907) provided an argument against empiricism from a foundationalist point of view: 'if experience alone can prove the truth of axioms, how can we know that they are true everywhere?'

There can be other arguments for apriorism, given from different perspectives. For example, from an ontogenesis of knowledge perspective, in which apriorism is identified with innatism, there is a known argument by C. G. Hempel, cited by Jerry Fodor (in M. Piatelli-Palmarini, 1979, p.380). The argument is as follows: suppose in a measurement the following pairs of numbers were obtained: (0, -1), (1, 0) and (2, 1). There are infinitely many possibilities for generalisation (for example,

$$y = x-1; y = (x-1)^3; y = (x-1)^{2n}\cos 1(1-x/2)\ for\ n = (1,2,\ldots)$$

If knowledge was a result of the individual's experiences then, in principle, all these generalisations would be equivalent. But there is an order of preference in the choice of the function model, which makes y = x - 1 the obvious choice. Fodor commented: 'One can call it simplicity, or the a priori order of functions, or innatism'.

1.1.2.1 Epistemology of the context of discovery: Poincaré and the French tradition in epistemology.

French philosophy of science is regarded as traditionally psychologistic and historicist (see, for example, Largeault, 1994). Accounts of actual processes rather than their rational reconstructions were attempted. In the field of the epistemology of mathematics, the works of Brunschwicg (especially *Les Etapes de la Philosophie Mathématique,* published in 1912), and the philosophical writings of Poincaré – published in articles in *L'Enseignement Mathématique* (see, for example, Poincaré, 1899, 1908a) and then in his well known books such as *Science et l'Hypothèse* (1906), and *Science et Méthode* (1908b) – have had an important influence. Cavaillès, Bachelard and Piaget were Brunschwicg's students.

The psychologism of the epistemologies of Poincaré, Bachelard and Piaget is evident. Bachelard's (1938) *La Formation de l'Esprit Scientifique* was a search for the 'psychological conditions of the progress of science'. Poincaré started an article by saying that the problem of the genesis of mathematical invention should inspire the most lively interest of a psychologist (Poincaré, 1908a). According to Poincaré, the 'context of discovery', or rather 'invention' (Poincaré was not a Platonist), was something worth studying because by reflecting on this process one could find reasons for errors in mathematics.

Although psychologistic, Poincaré's epistemology was nonetheless concerned with the origins of validity of our beliefs and not with the psychogenesis or history of scientific knowledge. Poincaré found these origins in the mathematician's synthetic a priori intuition and in his/her 'experience' or effective construction which allows him/her to verify if a postulated object exists. But intuition is fallible; a sudden illumination that has flattered the mathematician's aesthetic sense may turn false when submitted to the test of logical examination (1908a). Thus, in the construction of mathematical laws, intuition and logic interact; one in the invention process, the other in its verification.

There are, unexpectedly, many common features between the epistemological reflections of Poincaré and those of Dieudonné (1992), in spite of the link of Dieudonné to Bourbaki and its acclaimed logicism.