Chapter 3: Number and Arithmetic

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ABSTRACT

During the last decade several major shifts have occurred in the conceptualisation of mathematics as a domain, of mathematical competence as a goal for instruction, and of the way in which this competence should be acquired through schooling. This chapter begins with a summary of the general characteristics and principles underlying the ongoing world-wide reform of mathematics education. Afterwards it documents and illustrates how these general characteristics and principles permeate a major domain of the mathematics curricula for the elementary school, called 'Number and Arithmetic'. Five related topics within this domain are discussed, namely: number concepts and number sense, the meaning of arithmetic operations, mastery of basic arithmetic facts, mental and written computation, and word problems as applications of the numerical and arithmetical knowledge and skills.

The final section lists some remaining issues and tasks for further curriculum research and development in the domain of number and arithmetic.

1. INTRODUCTION

During the last decade, major shifts in the conceptualisation of mathematics as a domain, of mathematical competence as an instructional goal, and of the way mathematical competence should be acquired, have led to important changes in the content and the process of mathematics education. This is reflected in curriculum documents from several countries such as the *Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics 1989) and the *Mathematics Framework for California* (California Board of Education 1991) in the United States, *Mathematics Counts* (Cockcroft 1982) in the United Kingdom, the Dutch *Proeve van een Nationaal Programma voor het Reken/Wiskundeonderwijs op de Basisschool* (Treffers, De Moor & Feys 1989; Treffers & De Moor 1990), and *A National Statement on Mathematics for Australian Schools* (Australian Education Council 1990).

Most important with respect to mathematics as a domain is the shift from seeing mathematics as a large collection of concepts and skills to be mastered, to conceiving it primarily as a human sense-making and problem-solving activity. Concerning mathematical competence as an instructional goal, there is

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now general agreement that the ultimate objective of student learning is the acquisition of a mathematical disposition rather than a set of isolated concepts and skills. These shifts in the perception of mathematics and mathematical competence accompanied a change in the view of learning in general and of learning mathematics in particular, namely from a traditional conception as a passive and isolated absorption of bits of information and procedures provided by others to a view of learning as the active construction of knowledge and competence in a mathematical community (De Corte, Greer & Verschaffel 1996; De Corte, Verschaffel & Greer 1994; Kilpatrick 1994; Romberg 1994).

In the present chapter we document and illustrate how these general characteristics and principles underlying the ongoing world-wide reform of mathematics education permeate a major part of the mathematics curricula for the elementary school, namely 'Number and Arithmetic'. In doing so, we do not want to suggest that all current reform movements are unitary; we rather like to refer to them as a set of world-wide initiatives linked by a number of 'family resemblances'.

The first section summarises the major characteristics and principles of the current world-wide reform of the mathematics curricula, which are of great relevance for learning and teaching numeration and arithmetic in the elementary school. In the next sections these general features are illustrated and discussed with respect to five related topics within this domain of the mathematics curriculum, namely:

1) number concepts and number sense,
2) the meaning of arithmetic operations,
3) mastery of basic arithmetic facts,
4) mental and written computation, and
5) word problems as applications of the numerical and arithmetical knowledge and skills.

The final section lists some remaining issues and tasks for further curriculum research and development in the domain of number and arithmetic.

2. MAJOR CHARACTERISTICS AND PRINCIPLES OF THE REFORM DOCUMENTS

2.1 A mathematical disposition as the ultimate objective of mathematics education.

Careful analyses of expertise in a large variety of domains, including mathematics, have led to the identification of the crucial aptitudes involved in competent learning and problem solving. With respect to mathematics, there is nowadays a rather broad consensus that the major categories of aptitudes underlying skilled learning and problem solving are:

1) domain-specific knowledge,

2) heuristic methods,
3) metacognitive knowledge and skills, and
4) affective components like beliefs, motivations and emotions (see De Corte 1995; De Corte et al. 1994, 1996; Schoenfeld 1992).

Moreover, it has been argued that expertise in mathematics requires more than the mere sum of these four categories of aptitudes; those categories should be applied integratively and interactively. In this respect, the notion of 'a mathematical disposition' is useful to refer to the ability to apply of all those aptitudes integratevily. Moreover, implied in this dispositional notion is that – besides 'ability' – 'inclination' (defined as the tendency to engage in mathematical behaviour) and 'sensitivity' (conceived as feeling for, and alertness to opportunities for implementing the appropriate behaviour) are also crucial components of mathematical competence (De Corte et al. 1996), as described by the National Council of Teachers of Mathematics in the United States:

> 'Learning mathematics extends beyond learning concepts, procedures, and their applications. It also includes developing a disposition toward mathematics and seeing mathematics as a powerful way to looking at situations. Disposition refers not simply to attitudes but to a tendency to think and to act in positive ways. Students' mathematical dispositions are manifested in the way they approach tasks whether with confidence, willingness to explore alternatives, perseverance, and interest and in their tendency to reflect on their own thinking.'
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> (NCTM 1989, p.233).

This dispositional view of mathematical competence is reflected in the general goals of mathematics education as stated in the above-mentioned reform documents, which entails (much) more than the acquisition of a variety of domain-specific concepts, relations, rules, procedures and their applications. These goals include also the development of higher-order skills like the ability to explore, to conjecture, to reason, to reflect and to communicate mathematically, as well as the ability to use these cognitive and metacognitive skills effectively for solving non-routine problems. In addition, for each individual, a mathematical disposition involves developing a positive attitude toward mathematics, seeing mathematics as a powerful way of looking at situations, and becoming confident in one's own mathematical ability (see e.g., NCTM 1989; Treffers et al. 1989).

According to the reform documents, these higher-order goals should not be addressed separately through special teaching/learning activities. Instead, the specific content goals should be developed in such a way that these higher-order goals are realised simultaneously (NCTM 1989; Treffers et al. 1989). Or, as stated in the *Standards*, 'the curriculum should be permeated with these goals and experiences so that they become commonplace in the lives of the pupils' (NCTM 1989, p.5).

2.2 Constructive and authentic learning environments as a means.

Taking into account the preceding conception of what constitutes a mathematical disposition, new responses to the question how this general instructional goal should be reached have also been raised and to some extent already been materialised in new curriculum documents, instructional materials and textbooks. This section describes the main principles underlying this new approach (see also De Corte et al. 1996; Gravemeijer 1994; Kilpatrick 1994; NCTM 1989; Romberg 1994; Treffers 1987).

2.2.1 Learning mathematics is a constructive activity.

All reform documents start from the idea that learning mathematics is basically a constructive process. This idea is at odds with the view on which mathematics teaching in schools is still frequently more or less implicitly based, namely that learning consists in the rather passive absorption of knowledge gained and institutionalised by past generations. Considering learning mathematics as constructive means that pupils gather, discover, create mathematical knowledge and skills mainly in the course of some social activity that has a purpose. Consequently, mathematics classroom instruction should move away from the information-transmission model whereby the teacher is the supplier and the pupils are the passive recipients of mathematical knowledge and skills. Instead of being the main if not the only source of information, the teacher becomes a 'privileged' member of the knowledge-building community of the classroom who creates an intellectually stimulating climate, models learning and problem-solving activities, asks provocative questions, provides support to students through coaching and guidance, and fosters students' responsibility for their own learning.

2.2.2 The important role of meaningful and authentic contexts.

According to the reform documents, meaningful and authentic contexts should play a crucial role in elementary mathematics learning and teaching. New concepts and skills should first be encountered in challenging problem situations derived from real-life experiences or from exploring fascinating imaginary worlds. These situations can be presented in the form of a story, a drawing, a (dramatic) play, etc. Such meaningful contexts should be carefully selected and prepared to increase the probability that pupils' inventions and constructions about the mathematical concepts and skills to be learnt are appropriate and useful learning steps. However, meaningful and authentic contexts not only play an important role in the initial phase of the teaching/learning process. They must permeate it continuously to prevent mathematics from becoming separated from reality. Therefore, it is essential that the context problems represent the diversity, the complexities and the ambiguities of the problem situations pupils may encounter outside the mathematics class.

2.2.3 Progressing towards higher levels of abstraction and formalisation.

In the early stage of learning a particular numeration or arithmetic concept or skill, pupils use their intuitive knowledge and informal skills to make sense of and to solve the context problems they are facing. But these intuitive notions and informal approaches have, of course, their restrictions, such as their lack of precision, efficiency and generalisability. Therefore, they should be transformed into more efficient, more formal and more abstract mathematical concepts and skills. In this transformation process, which involves activities of progressive schematisation, abbreviation, internalisation and generalisation of informal and context-bound mathematics, a crucial role is played by carefully chosen mathematical models and tools. Manipulative, visual models, schemes, and diagrams can be used as scaffolds fulfilling this bridging function between children's intuitive notions and informal strategies, on the one hand, and the concepts and procedures of formal mathematics, on the other. Pupils themselves should as much as possible play a role in developing and refining these models and tools.

2.2.4 Learning through social interaction and co-operation.

As already suggested above, doing and learning mathematics should not be conceived and practised as a purely solo activity.

Whole-class teaching and individual work must be combined with co-operative learning in small groups and classroom discussions. Social interaction is considered essential because of the importance in learning and doing mathematics of exchanging ideas, comparing solution strategies, and discussing arguments. Of special significance in this regard is that interaction and collaboration mobilise reflection, which is considered as the basic mechanism for attaining higher levels of abstraction and internalisation.