CHAPTER 1

INTRODUCTION
FOUNDATIONS OF MATHEMATICS

1. INTRODUCTION

Foundational questions in mathematics were born with Hilbert, but foundational programs existed before him. Arithmetization of analysis and arithmetization of algebra (for Kronecker) antidate Hilbert's idea of axiomatization. While Frege was struggling with the logical concept of number as the extension of a concept and while Cantor (and Dedekind) imagined infinite (transfinite) extensions of the ordinary number concept. Kronecker was busy devising a general arithmetic that would arithmetize mathematics without transcending the realm of the algebraic. The so-called foundational crisis did affect only the logicist program, comforting in a sense the arithmetical program. It is that program that Hilbert wanted to pursue with other means in order to rescue set theory from its logico-paradoxical consequences.

Logic is reponsible for the paradoxes of set theory and logic should be able to solve them : Frege's *Grundgesetze der Arithmetik* (1893) is contemporary to Cantor's set theory and it is intended as a formalization of arithmetic. Frege's fifth axiom also called the axiom of unlimited comprehension read

$$\exists x \forall y (y \in x \leftrightarrow P(y))$$

for an arbitrary predicate P : a contradiction $x \notin x$ readily ensues when one substitutes x for y. Russell's solution is well known, type theory in its simple or ramified version. Type theory is built upon a hierarchy of predicates (or functions, functions of functions, etc. over individual variables) which are ordered by their type or rank in the theory of simple types and moreover by their order or level in the theory of ramified types. The first type (noted 0) is the type of individuals, the second one (noted (0)) is the type of sets of individuals and of relations (predicates) or operations (functions) over individuals and so on. Besides the hierarchy of types, one has a corresponding hierarchy or orders which allows for the definition of order of quantified functions : for example, $\forall \varphi (\psi(\varphi, y))$ where φ is a first-order function will have the order two, since φ is bound and ψ occurs here as a functional (a function of a function). In other words, the notion of order was needed for the hierarchy of bound variables of a function of a given order.

But the Russellian solution, like Frege's program, was rooted in the logicist program, the reduction of arithmetic and mathematics to logic through the axiomatization of the notion of (natural) number. However, the introduction of the axioms of infinity and choice, and the axiom of reducibility in the theory of ramified types, which postulates that one can always find a predicative function of order r or lower order formally equivalent to a function of order r, all those non-logical axioms have undermined the logicist thesis, since the axioms of choice and infinity are not logical by their nature and the axiom of reducibility is purely artificial, as Russell finally admitted. It is not out of order to suggest that the type theory of *Principia Mathematica* deserves only historical interest.

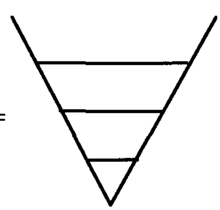
E. Zermelo's solution (1908) to the paradoxes, axiomatic set theory, was bound to follow a more successful course. Instead of Frege's axiom of unlimited comprehension, Zermelo introduced an axiom of limitation, the axiom of separation <*Aussonderungsaxiom*> which reads

$$\forall x \exists y \forall z (z \in y \leftrightarrow z \in x \wedge A(z))$$

meaning that for any set x, there is a set y the members of which are those members of x which satisfy the formula A (or the property X in a second-order formulation). Zermelo's axiomatic set theory completed by Fraenkel (and designated by Z - F) has become the foundational theory of contemporary mathematics. We shall see later why intuitionism and category theory cannot cope with set theory as a foundational framework.

The most common formulation of Z - F rests upon the cumulativee rank structure

$V =$



V_γ (for an inaccessible ordinal γ)

$V_\alpha = \cup_{\beta<\alpha} V_\beta$ (for $\forall$ limit - ordinals)

$V_{\alpha+1} = V_\alpha \cup P(V_\alpha)$ (for $\forall$ ordinals)

$V_0 = \varnothing$

The cumulative structure embodies the generation of the hierarchy of sets in the elementary operations of union and power set starting with the null set as first rank $V_0 = \phi$. Here α is an arbitrary ordinal, its successor is defined by

$$V_{\alpha+1} = V_\alpha \cup P(V_\alpha) \qquad \forall \alpha$$

and V_γ is

$$V_\gamma = \bigcup_{\beta<\gamma} (V_\beta) \qquad \gamma \text{ limit ordinal.}$$

The structure is open, γ can be considered as a (strongly) inaccessible ordinal, that is $\gamma > \omega$, ω being the first non-finite ordinal and γ is not accessible by the union or the power set operations of smaller ordinals (or cardinals). In order to interpret mathematical theory in Z - F, one has only to choose a sufficiently high ordinal in the hierarchy : the von Neumann-Bernays-Gödel set theory and category theory are readily embeddable in Z - F.

2. PHILOSOPHY OF MATHEMATICS AND FOUNDATIONAL RESEARCH

Philosophy seems to be left behind in the area of problems we have just evoked. Logicism is a philosophical thesis which is no more a currency nowadays. The reduction of mathematics to logic has not given rise to any interesting mathematical result. Hilbert's program of combinatorial foundations did meet with some measure of success, insofar as an enlarged finitism has not been put to death by Gödel's results. Kreisel has shown that a modified Hilbert's program is still viable, provided that one welcomes, beyond the finite combinatorics of concrete objects (symbols), abstract objects like the functions and functionals of Gödel's *Dialectica* (1968) interpretation by which an "abstract" proof of the consistency of arithmetic is made possible. Formalism professes agnoticism towards infinite sets, a skeptical attitude which is not philosophically appealing. And nominalism or pragmatism, defended for example by Putnam and Quine at one time, are programs of limited appeal, since thay are mainly concerned with the elimination of abstract entities or useless fictions. What is needed is a more radical program which shows how transfinite cardinals, among others, are harmless idealizations. The other philosophical theses, from conventionalism to empiricism or materialism or even structuralism, are mere variations on two major themes which I want to exlore now, realism and constructivism.

Realism and constructivism, besides particular theses on the status of mathematical objects, are the only foundational options which have a direct connection with mathematical practice; they are global theories of mathematical activity. The realist is often naïve, that is, she takes most of the time a naïve stand concerning foundational questions : she will say, for example, that the mathematician feels that she discovers mathematical reality and that what she constructs seems artificial to her compared to the "naturalness" or depth of the "pure" discovery; she also wants to believe in axioms, as a kind of abridged dogmas. Obviously, even the realist with her well-established opinions or convictions, cannot dispense with analysis. The mathematician here is not any more competent than the epistemologist or, as I prefer to say, the epistemologician, the logician of true knowledge, in formulating the "good" theory of mathematical practice and as a working mathematician she is not granted any kind of priviledge, for she does not necessarily enjoy critical distance, theoretical retreat and conceptual amplitude. In other words, propinquity does not breed knowledge. The fact is that the working mathematicians are in general realist and naïve at that for many among them. But there are not all naïve to the same extent. Realism or Platonism (in a Gödelian sense) supposes that the ideal objects of mathematics are similar to the objects of sensation and that they are perceptible in an intellectual intuition. Such a conception has to do with a metaphysical option more than with the analysis of mathematical practice. Constructivism holds that mathematical objects are mental (that is, linguistic) constructions of a linguistic agent (the creative subject) who creates freely the mathematical universe. That freedom is nevertheless limited to effective procedures. Infinite totalities, illegitimate proof methods like the excluded third, indirect proof or *reductio ad absurdum* are proscribed while explicit construction methods are emphasized to enhance the security, <*Sicherheit*> as Hilbert put it, of mathematical work. Between these two poles of realism and constructivism, foundational research has to find a sure philosophical course.

3. MOTHER-THEORIES

I use the expression mother-theories in Bourbaki's sense of mother-structures or fundamental structures. Mother-structures are the topological, algebraic and order structures. As mother-theories in the foundations of mathematics, we have set theory, intuitionism and category theory.

3.1. Set theory

I shall not describe axiomatic set theory or Z - F in detail, but rather stress the main results and their foundational consequences. The two main results in Z - F are certainly Gödel's proof of the consistency for the continuum hypothesis and the axiom of choice and Cohen 's proof of independence —twenty-five years separate the two proofs (1938-1963). The continuum hypothesis states that

$$2^{\aleph_0} = \aleph_1$$

that is, the power set of a denumerable set $\aleph_0$ has the cardinality of the next higher non-denumerable cardinal $2^{\aleph_0}$. The Generalized Continuum Hypothesis (GCH) for ordinals σ

$$\forall\sigma\left(2^{\aleph_\sigma} = \aleph_{\sigma+1}\right)$$

is a natural extension. There are many equivalent statements of the axiom of choice; one of them is "the product of a family of non-void sets is non-void". Gödel's proof rests essentially on the notion of constructibility anchored in the cumulative rank structure. Let L denote the constructible hierarchy; then we have

$$L_0 = \phi$$

$$L_\alpha = \bigcup_{\beta<\alpha} L_\beta$$

and

$$L_\alpha = \bigcup_{\beta<\alpha} \Pi\left(L_\beta\right)$$

where $\Pi(L_\beta)$ means the set of all definable subsets of L_β. The class of constructible sets is then defined by

$$L(x) = \exists\alpha\{On(\alpha) \wedge x \in L_\alpha\}$$

and the axiom of constructibility

$$V = L$$

for V the set-theorectic universe. The constructible hierarchy is thus generated by the iteration of predicative definitions (definable subsets) up to an arbitrary ordinal : the generation of constructible sets is thereby isomorphic to the generation of ordinals. Notice that constructible here does not have a constructivist meaning, since is not known how the ordinals are found. In order to construct a model for L, it is enough to verify that the class L of constructible sets satisfies the axioms of Z - F plus the axiom of foundation which states that any non-void set contains an element which has no element in common with that set. Gödel's constructible model is a standard interior model. There exists a denumerable minimal model M which is isomorphic to any standard model N and for which $V = L$. Thus, if one wants to prove, as Cohen has shown, that Z - F plus the continuum hypothesis and that Z - F plus the axiom of choice do not imply $V = L$, one has to go beyond the model M and suppose $M \subset N$; the model N is built from M by injecting new sets $a \leq \omega$ such that $a \in N$, but $\neg(a \in M)$. Those new sets are called generic since they are defined collectively (by gender) and not individually. Since a is not in M, a is not constructible. The introduction of generic sets is controlled by forcing conditions which provide the new (finite) information necessary to force the injection of generic sets. For the notion of forcing conditions, one has

$$C \mid\!\vdash \neg\neg a \not\leftrightarrow C \mid\!\vdash a$$

for C a forcing condition and $\mid\!\vdash$ the forcing symbol. We see that the reinterpretation of logical connectives and quantifiers required by that notion of forcing is not too far from the intuitionistic interpretation of negation which forbids $\neg\neg a \rightarrow a$.

One of the dominant features of more or less recent set theory is the hunt for large transfinite cardinals or strong axioms of infinity. Measurable cardinals, compact cardinals, strongly compact cardinals, ineffable cardinals, and so on, have not yielded the desired result, a positive or negative answer to the continuum hypothesis or the justification of its many consequences like, for example, Martin's axiom. Other researches have put the emphasis on infinite combinatorial principles (Jensen). One such combinatorial principle of old vintage is the infinite Ramsey's theorem stating

$$k \rightarrow (\lambda)^\eta_\mu$$