CHAPTER 3

THE CONSISTENCY OF ARITHMETIC REVISITED

1. INTRODUCTION

The consistency problem was raised by Hilbert as a main problem in his famous list. Hilbert formulates his second problem in terms of the non-contradiction of the arithmetical axioms which are nothing else than the elementary arithmetical operations plus the axiom of continuity (see Hilbert, 1925, p.300). The last axiom, Hilbert says, can be split into two simpler axioms, the Archimedean axiom and the (syntactic) completeness axiom which he introduced in order to provide an arithmetical model of Euclidean geometry, thus proving its consistency. But consistency of arithmetic needs a direct proof <*ein direkter Beweis*> that would lead from a proof for the consistency of elementary arithmetic to a (finitist) proof of existence of the continuum, classical analysis and Cantor's tranfinite ordinals (with the exclusion of the totality of alephs). The direct way <*ein direkter Weg*> is a progression from elementary arithmetic of natural numbers N to the rationals Q through the integers Z to the real numbers R. The progression is the one that Kronecker in his *Über den Zahlbegriff* (1887a) had shown to proceed from the concept of number alone in his general arithmetic. This « arithmetic continuation » as I would like to call it, is the core of Kronecker's programme and Hilbert is seen here to continue it with logical means, *i.e.* the axiomatic method which Hilbert defines as a finite number of logical inferences from axioms (Hilbert, 1935, p. 301). Where Kronecker used purely arithmetic methods, for example, congruence relations and polynomial equations in his theory of forms, Hilbert introduced logical operations that are supposed to take over and go beyond arithmetic towards analysis and transfinite set theory, but in a finitist metamathematical framework. Thus logic is but a replica of general arithmetic or its continuation by other means. In order to state more fully the problem, one should add the following quotation taken from Hilbert notebooks and dated around 1905 (following M. Hallett, 1995, p. 152)

> Though the Archimedean and my completeness axioms [for Euclidean geometry or the reals respectively], the ordinary continuity axiom is divided into two completely different components. Moreover, with my completeness axiom, not one infinite process is demanded, but we have only a finite number of finite axioms, just as Kronecker demands.

My contention is that one cannot understand precisely Hilbert's intent if what I called Kronecker's programme is not taken into account and if Hilbert's programme of a finitist foundation for mathematics is to be regarded as meaningful at all. Gödel, who, apparently, has been reluctant to admit the (total) failure of Hilbert's programme, admits that an *internal* finitist proof of consistency of arithmetic is not excluded (see Gödel, 1966).

In my view, the problem of consistency must be replaced in that context. Hilbert's proof for the consistency of Euclidean geometry was grounded (on the model) of the arithmetic of the reals. If analysis is to be proved consistent, alongside with set theory, that is Cantor's second number class (excluding the hierarchy of all powers), one must begin with finite arithmetic *à la* Kronecker. I call this arithmetic Fermat-Kronecker arithmetic, since Fermat's infinite descent replaces the induction postulate and Kronecker's indeterminates <*Unbestimmte*> play the rôle of variables in the general arithmetic <*allgemeine Arithmetik*> of forms or (homogeneous) polynomials. I assume that the classical arithmetic Hilbert has in mind was classical number theory, not Dedekind-Peano arithmetic or set-theoretic arithmetic.

Kronecker's general arithmetic consists not only of natural and rational number systems, but also of abstract algebra, fields or domains of rationality and fundamental constructions in arithmetic-algebraic geometry. The model theory of first-order structures that are decidable, algebraically closed fields, real-closed fields are also comprised in that general arithmetic which I would call polynomial arithmetic.

In his remarks of 1966, Gödel speaks of ω-consistency in terms of outer consistency. I take the expression to mean the totality of natural numbers, thus actual infinity of the set N that is an ω-model of N which is also complete as Tarski has shown in 1933; ω-consistency can be secured only by ε_0-consistency. Gentzen, Ackermann, Gödel and others will need transfinite induction over the transfinite ordinals, the ω's of Cantor's second number class. Gödel's *Dialectica* (functional) interpretation replaces transfinite induction by induction over all finite types beyond the type of natural numbers. Although Kreisel says in his 1976 paper that Gentzen had used a version of infinite descent in his proof, Gentzen calls it a disguised form of complete induction comparable to the Euclidean algorithm; I maintain in the following that from a constructivist viewpoint, infinite or indefinite descent, as Fermat calls it, is not equivalent to complete induction — the equivalence would require a double negation over an infinite set, an operation clearly forbidden by constructivist or intuitionistic standards. Hilbert's programme has not succeded, not much because of Gödel's incompleteness results, but because Hilbert had hoped to jump beyond finitism despite his own mathematical convictions. The formal extension through metamathematics, that is logic and the epsilon symbolism for the transfinite choice function, loses sight of the finite point of view <*finiter Standpunkt*>. Despite his numerous attacks against Kronecker as the <*Verbotsdiktator*>, Hilbert acknowledges his debt to Kronecker in his 1930 « Die Grundlagen der elementaren Zahlenlehre » :

> Kronecker has clearly formulated a conception which he has repeatedly illustrated; his conception corresponds essentially to our finitist stand (p. 487),

but Hilbert quickly adds that Kronecker's error had been the banishment of infinitary (transfinite) proof methods, for example, excluded middle and presumably analytic continuation. But again, transfinite statements are ideal and can be eliminated and Hilbert has not been able to resdescend from the paradise Cantor had created for him! My hypothesis, that is my conceptual reconstruction of history — which has no historical pretension — is that Hilbert was divided between Cantor' paradise and

Kronecker' solid ground and that going away from Kronecker he lost ground and grip of the consistency problem.

2. FINITISM

Hilberts programme can be modified, as Kreisel has suggested, it can be extended as Gödel supposed and it could be relativised in various directions as Feferman and Nelson have proposed. I have attempted rather to radicalize Hilbert's programme by founding it on Kronecker's programme. When Hilbert in his talk « Ueber das Unendliche » (1926) explains that, from the finitist stance <*finiter Standpunkt*>, there are two kinds of formulas in mathematicss, the ones that correspond to finitary statements and the ones that refer to ideal (meaningless) structures, he is just translating Kronecker's language of a general arithmetic whith its (indeterminate) extensions — which cover ideal elements — into metamathematics or the theory of proofs he wants to formulate. But the extra-arithmetical operations of logic are as meaningless as the algebraic quantities outside domains of rationality and if arithmetic alone is internal — algebra is formal or external — the formal system of logical operations will have the limited function of an extension of arithmetic, provided that such an extension is consistent, that is, once the ideal structures (or indeterminates) are eliminated, the validity of the logical laws of the primitive domain of arithmetic, in Kronecker's terms, the arithmetic of the natural domain of rationality is preserved. One sees immediately the close parallelism between Kronecker's programme and Hilbert's programme. The relationship is so striking that one could suppose that Hilbert is constantly inspired, consciously or not, by Kronecker's arithmetical constructivism.

The concrete objects that are going to replace integers in Hilbertian metamathematics are the signs and symbols of a finite combinatorics which is the formal counterpart of internal <*inhaltliche*> arithmetic or arithmetic with arithmetical content. At the beginning is the sign, this is Hilbert's philosophical motto as early as 1902. On a finitary basis, says Hilbert, existing mathematical theories can be formalized by the joint construction of logic and arithmetic. The resulting "arithmetical logic", as I call it, contains an internal (metamathematical) logic, which goes beyond the formal proof of ordinary mathematics, and leads to a proof of the non-contradiction of mathematics, since the object of metamathematics is the totality of proofs in usual mathematics. It is clear from Hilbert's pronouncements that there is a direction forward from arithmetic to set theory and analysis and while the ground arithmetical logic produces new axioms, formal logic only proceeds to the derivation of new theorems from existing axioms. Finitary logic suffices to warrant the intuitive validity of elementary arithmetic, but traditional logic should be able, Hilbert assumes, to extend that validity beyond elementary arithmetic. Hilbert then defines connectives and quantifiers accordingly using a transfinite choice function $\varepsilon(A)$ which associates an object to each predicate and a number to each function; thus, the universal quantifier is defined by a choice function which cannot find a counterexample to a given predicate (or to the image of a given function). Hilbert adds the Aristotelian axiom for the existential import of the universal quantifier and the principle of excluded third which means that negation of the universal quantifier implies the existence of a counterexample.

Although the choice function is not constructive, Hilbert believed that its iteration or reiteration a finite number of times secured the finite character of the procedure and that a consistency proof along those lines was certainly possible. Ackermann, as is well known, has succeeded in giving a consistency proof of arithmetic with the Hilbertian ε-substitution method (and with transfinite induction).

Hilbert's programme has failed because of Gödel results, but more importantly it has failed because it has deviated too far away from Kronecker's original programme. Kronecker had resisted the infinitist temptation by keeping close to arithmetic and if Hilbert has yielded to the temptation, it is due to his submission to the (presumed) existence of ideal elements or to the formal definiteness of indeterminates, as one could say, the final elimination of which he could not achieve in his attachment to Cantor's paradise. Hilbert's formalism or rather the formal extension of finitist mathematics is but the non-finitist enlargment of the finitist position <*finite Einstellung*> and the dissolution of absolute consistency in relative consistency. It is not surprising, in retrospect, that it is the infinite induction of set-theoretic (Peano) arithmetic which is the heart of the matter. Hilbert in 1930 is still reproaching Kronecker with is rejection of infinitary proof methods and it is an ironic dramatic surprise that Gödel published, a year after Hilbert's paper (1930) his incompleteness proof for Peano arithmetic using a method of proof which can be said infinitary, since it uses Cantor's diagonal procedure on the infinite set of natural numbers.

Cauchy's product or the convolution product, which we call after Kamke (1965), Cauchy's diagonal, does not lead outside the realm of the finite. The expression

$$\sum_0^n c_n x^n = \left(\sum_0^n a_n x^n\right)\left(\sum_0^n b_n x^n\right)$$

with $c_n = a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_n$ defines a constructive procedure, the finite summation of integer coefficients. Polynomials of finite degree are polynomial functions of the form

$$f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$$

and are the finite support of infinite power series. It is that theory of polynomials which is at the center of Kronecker's theory of forms or homogeneous polynomials and his general arithmetic is at the same time a theory of algebraic divisors. The arithmetic of modular systems, call it general modular arithmetic, as exposed in (Kronecker, 1882), is the central thema.

In Kronecker's words, his theory of forms, which is a kind of arithmetization of algebra, makes essential use of indeterminates <*Unbestimmte*> in its method of association-elimination of forms. The aggretates, as he says, are domains of rationality to which can be associated forms in different genera and species. A general divisibility theory reproduces in a like manner the decomposition of (composite or associated forms) in irreducible factors. Thus, Kronecker's general arithmetic is a calculus of indeterminates associated to the arithmetic of integers and Kronecker's programme consists mainly in the reduction of algebraic quantities to a polynomial arithmetic. The arithmetical theory of algebraic quantities can in fact be summarized as the theory of entire functions with integer coefficients including the theory of entire rational functions in any domain of rationality. One knows that an entire function — taking all its finite values — which is not a polynomial is a transcendantal function by definition, that is, it is not algebraic. But Kronecker could adjoin indeterminates or indeterminate integers to a domain of rationality or a field to take care of those purely negative entities, the transcendantal numbers. The arithmetical existence of algebraic entities on the other side enjoys conceptual determinations, equivalence properties like congruence and operations like substitution in an equational calculus that allows for association and adjunction in the extension of the domain of arithmetic. The concept of content of forms insures moreover that the process of association does not transgress the bounds of the domain of rationality. Such conservative extensions of arithmetic cover all of algebra and the theory of forms can finally be seen as a theory of generalized integers apt to encompass arithmetic within the reach of the finite. Fermat's method of descent is then called for as the proprer induction principle in the finite arena of polynomial arithmetic.

3. FERMAT'S DESCENT

Let us call Kronecker's general arithmetic KA. That arithmetic encompasses Z, the arithmetic of integers or elementary arithmetic. If one can show that KA is consistent, then Z will be shown consistent, since it is embededd in KA. KA must contain an induction principle that enables the decomposition of complex formulas (equations) into simple elementary ones, just as arithmetic must exhibit its self-consistency by elementary means $0 \neq 1$, thus reflecting the elementary logical fact of consistency $T \neq F$. The induction process appropriate for that purpose is Fermat's infinite descent which is also a central method of proof in pure number theory (from Fermat to Kummer to Weil).

Fermat says of infinite descent that it is an *apagogê eis adunaton* or *reductio ad absurdum*. Fermat also states in his 1670 commentary on Diophantus :

> Eodem ratiocinio dabitur et minor istâ inventa per viam prioris, et semper in infinitum minores invenientur numeri in integris idem praestantes. Quod impossibile est quia, dato numero quovis integro, non possunt dari infiniti in integris illo minores (Fermat, 1891).

I translate the last quotation as :

> By the same calculation it is supposed that a smaller number is found in a descending procedure and that one can always find numbers smaller than the preceding one *ad infinitum*, which is impossible, since that, for an arbitrary integer, there cannot be found an infinity of smaller ones in integers.

Let us remark that the method of infinite descent can be applied to a variety of problems, starting with the proof of the irrationality of $\sqrt{2}$ or the impossibility of

$$x^4 + y^4 = z^2$$