CHAPTER 4

THE INTERNAL CONSISTENCY OF ARITHMETIC WITH INFINITE DESCENT

1. INTRODUCTION

A proof of the consistency of arithmetic without the induction postulate, but with infinite descent is given in the following. No use is made of transfinite induction and "internal" means that infinite descent will be shown to be self-consistent. I call this arithmetic with infinite descent Fermat arithmetic (FA) to contrast it with Peano arithmetic (PA) (see Gauthier, 1989). The main idea is to translate logic into arithmetic via a polynomial interpretation with Kronecker's indeterminates and is thus an attempt at the arithmetization of logic in the line of what can be called "Kronecker's programme". The logic is constructive, that is it has all the intuitionistic features plus some constructive (local) characteristics to be described below. Fermat arithmetic is minimal in the sense that it is sufficient for (elementary or constructive) number theory and algebra up to (some important part of) algebraic or arithmetic geometry. André Weil has stressed the import of Fermat's infinite descent and Kronecker's arithmetical theory of algebraic quantities in the making of modern mathematics, but the constructive nature of such proof methods has not been generally recognized by logicians. Rather, logicians in general have tended to assimilate infinite descent and complete induction on the one side and favor Dedekind's transcendental method over Kronecker's algorithmic approach on the other[1] (see Edwards, 1987).

From a (classical) logical point of view, infinite descent is identified with the least number principle

$$\exists x Ax \rightarrow \exists x[Ax \wedge \forall y(y < x \rightarrow \neg A(y))]$$

for a formula A and y different from x with no occurence in A. This principle can be obtained from the principle of complete induction

[1] Exceptions are found mainly among mathematicians. Poincaré, Mordell, Weil not to mention more recent workers in algebraic geometry have all used infinite descent as a ''more'' effective method of proof than complete induction. As for Kronecker, Weil has made clear that he is the true originator of algebraic "arithmetic" geometry. The present work could be seen as a vindication of "Kronecker's programme" of a general arithmetic <*allgemeine Arithmetik*> in the foundations of mathematics for which Kronecker claimed internal truth and consistency <*innere Wahrheit und Folgerichtigkeit*>.

$$\forall x[\forall y(y < x \to Ay) \to Ax] \to \forall xAx$$

which is deducible from Peano's induction postulate

$$\forall x[A0 \wedge \forall x(Ax \to ASx)] \to \forall xAx.$$

Transfinite induction substitutes ordinals σ for natural numbers in the following schema

$$\forall\sigma[\forall t(t < \sigma \to A(t,x) \to A(\sigma,x))] \to \forall\sigma A(\sigma,x)$$

with the limit

$$\lim_{n\to\infty} \omega^{\omega^{\cdot^{\cdot^{\omega}}}}\}^{n} = \varepsilon_0 .$$

Transfinite induction has been used by Gentzen in his proof of the consistency of arithmetic and Ackermann could not but invoke it in his own proof (1940).

From a different point of view, Nelson (1986) offers a predicative or bounded version of the least number principle. From

$$min\, x_1 \ldots x_r A \equiv A \wedge \neg\exists y_1 \ldots y_r(y_1 \leq x_1 \wedge \ldots \wedge y_r \leq x_r$$
$$\wedge (y_1 \neq x_1 \vee \ldots \vee y_r \neq x_r) A x_1 \ldots x_r [y_1 \ldots y_r])$$

where the y's do not occur in A and are all different from the variables x, the principle simply states

$$\exists x_1 \ldots \exists x_r A \to \exists x_1 \ldots \exists x_r min\, x_1 \ldots x_r A$$

a metatheorem which is proven within predicative arithmetic. Buss (1986) has shown how $\sum_i^b$*–LMIN* axioms are equivalent to the $\prod_i^b$*–PIND* or corresponding bounded induction axioms. But Nelson's arithmetization of (classical) logic stops short of a consistency proof for arithmetic with infinite descent, although there is a proof of the self-consistency of Robinson's theory Q using the Hilbert-Ackermann consistency proof with quantifier elimination. Our aim is to obtain self-consistency for a larger theory, FA, with constructive means in the polynomial interpretation.

We look at logic as arithmetical logic, that is logical formulas are interpreted as polynomials and constants as arithmetical operations. This last point was emphasized by Ackermann (1940)[2]. The introduction of Hilbert's ε-symbol and its subsequent

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[2] Skolem's quantifier-free primitive recursive arithmetic and Goodstein's equational calculus foreshadow arithmetical logic, but explicit use of complete induction in Skolem (1970) is alien to Fermat arithmetic

elimination in proofs of consistency (cf. Herbrand and Ackermann) for the predicate calculus and pure number theory have also inspired the way we treat "effinite" quantification through reduction by infinite descent.

Finally, it should be noticed that infinite descent has entered axiomatic set theory by the front door. The axiom of foundation formulated by von Neumann

$$\forall x\{x \neq \varnothing \rightarrow \exists y(y \in x \wedge y \cap x = \varnothing)\}$$

comes from what Mirimanoff (1917) called ordinary sets <*ensembles ordinaires*> which generate only finite descents. A sequence of elements $e_1 \ni e_2 \ni e_3 \ldots$ of a set E stops when one descends to an indecomposable element, that is $\varnothing$, also called « core » by Mirimanoff. The axiom of replacement also formulated by von Neumann (1961) (inspired by Fraenkel) in the form

$$Func(f) \rightarrow f''x \in V$$

which means that if x belongs to the set-theoretic universe V, its image also belongs to V. It is easily seen that we have here the cumulative hierarchy. Mirimanoff had already the three operations (or postulates) for the cumulative hierarchy, union, power set and replacement which he explains as :

> If a set $(a,b,c,\ldots)$ exists, then any equivalent set $(E,F,G,\ldots)$ exists, where $E,F,G\ldots$ are existing (distinct) ordinary sets.

Takeuti has attempted a justification of transfinite induction by resorting to infinite descent in his (1975), but von Neumann (1961) under the impulse of Mirimanoff's infinite descent could already introduce ordinals through transfinite induction

> Every ordinal is the set of ordinals preceding it (« *Jede Ordnungszahl ist die Menge der ihr vorangehenden Ordnungszahlen* »).

Together with the axiom of replacement it was the birth certificate of the cumulative hierarchy.

Historically, transfinite induction was introduced by Hausdorff as complete induction on Cantor's transfinite ordinals in their normal form (polynomial)

$$\phi = \omega^{\mu} \nu_0 + \omega^{\mu-1} \nu_1 + \ldots + \nu_{\mu}$$

with decreasing finite powers. In that context, transfinite induction is precisely infinite descent extended to transfinite ordinals. We shall see, however, that infinite descent over the integers is sufficient for arithmetic.

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while the unlimited (unbounded) substitution of number variables by definite numerals in Goodstein amounts to complete induction. See Goodstein (1951).

2. LOGIC

The logic is presented in a sequent calculus which is minimal, with no structural rules but with new notions, *i.e.* two new connectives, local negation and local implication and a new quantifier called the "effinite quantifier". The basic concept "sequence" is divided in two, finite sequences which are sets and effinite sequences which are not. There are no infinite sequences. An effinite sequence is open-ended, that is, it has a pre-positional bound, *e.g.* 0, but no post-positional bound, *e.g.* ω. An effinite sequence is somewhat like Brouwer's infinitely proceeding sequences without any pre-assigned limit. When an effinite sequence has post-positional bound, it becomes an initial segment, *i.e.* a set. Though it is minimal, the radical logic we are devising aims at providing a natural framework for arithmetic, that is constructive theorems of number theory, *e.g.* Euclid's theorem on the infinity of primes. In a way, our logic is a finite probe for the concept of infinity. All notions are meant to be local and the logic itself is a "local logic".

The universe consists of the effinite sequences of natural numbers which we call the arithmetical domain D.

Remarks : This notion of domain has some similarity whith the domains <*champs*> of Herbrand's Fundamental Theorem where « the necessary and sufficient condition for a proposition not to have property B is that it be false in some infinite domain » (Herbrand, 1971). However, we do not need here Herbrand's notion of order, since a post-positional bound on an effinite sequence makes a (finite) set out of it.

2.1. Syntax

Our first-order language $L(T)$ for our first-order theory T has an effinite supply of atomic symbols :

1) letters (capital and small) for formulas (and sentences), $A, B, C, \ldots$ together whith their punctuation signs, points, commas, parentheses brackets, etc.

2) letters for variables $x_1, x_2, \ldots, x_n$,

3) predicate letters p_j^n and the predicate symbol =

4) functions letters f_j^n —when f is 0-ary, we consider it as a constant

5) the connectives $\wedge$, $\vee$, $\neg$, $\rightarrow$

6) the quantifiers $\forall$, $\exists$ and Ξ.

The terms consist exclusively of :

1) variables

2) sequences composed of terms and functions letters, *e.g.* $f_j^n\, t_1, \ldots, t_n$ —for the terms $t_1, \ldots, t_n$.

Formulas or wffs consist exclusively of :

1) atomic formulas composed of terms and predicate letters, *e.g.* $p_j^n t_1, \ldots, t_n$ for the terms $t_1, \ldots, t_n$

2) any wff consisting of formulas composed of connectives and quantifiers.

Remarks : Sentences are closed formulas, *i.e.* formulas are "open" sentences where variables occur free, that is, are not quantified upon. An instance $A(t_1,\dots,t_n / x_1,\dots,x_n)$ of a formula A is the result of subtituting terms t for the free occurrences of a variable x.

I adopt the standard formulation of the sequent calculus (see, for example, Girard (1987)). A sequent is an expression $\Gamma \square \Delta$ where Γ and Δ are finite sequences of formulas; Γ is the antecedent, *e.g.* $A_1 \wedge \dots \wedge A_n$ and Δ the succedent, *e.g.*, $B_1 \vee \dots \vee B_n$ with the interpretation $(A_1, \wedge \dots \wedge A_n) \to (B_1 \vee \dots \vee B_m)$.

2.1.1. Axioms

The system of *LL* (Local Logic) has for only axiom the identity or equality axiom

$$\textit{Axiom} \quad A \square A$$

for A an arbitrary formula. Since we do not have structural rules (see below), we take as axioms all formulas of the form

$$\Gamma, A \square A, \Delta \qquad \text{(for the weakening rule).}$$

2.1.2. Logical rules

Logical rules are expressed in the sequent calculus with a left-right symmetry while in a system of natural deduction, this symmetry is replaced by the *intelim* rules (introduction and elimination rules). The bar indicates that the sequent of the conclusion under the bar has been obtained from the sequent of the premiss by the given rules. Since our system is a system of local logic (with minimalist and intuitionistic properties), in practice we can consider only sequents $\Gamma \square \Delta$, where Δ consists of a unique formula.

The logical rules are the following :

Conjunction

$$\frac{\Gamma \square A, \Delta \quad \Gamma \square B, \Delta}{\Gamma \square A \wedge B, \Delta} \wedge r$$

$$\frac{\Gamma, A \square \Delta}{\Gamma, A \wedge B \square \Delta} \ell 1 \wedge \qquad \frac{\Gamma, B \square \Delta}{\Gamma, A \wedge B \square \Delta} \ell 2 \wedge$$

Disjunction

$$\frac{\Gamma \square B, \Delta}{\Gamma \square A \vee B, \Delta} r2 \vee \qquad \frac{\Gamma \square A, \Delta}{\Gamma \square A \vee B, \Delta} r1 \vee$$