CHAPTER 5

FROM KRONECKER TO BROUWER

1. INTRODUCTION. CANTOR

There is no need to recall the polemics between Kronecker and Cantor, simply because it did not exist. H. Edwards (1987) has recently shown that if there has been any quarrel between the two, it has not been the virulent one some had evoked. The intransigent Kronecker, a <*Verbotsdiktator*> in Hilbert's words, is supposed to have been a fiend of Cantor, whom he would have called a youth perverter <*Verderber der Jugend*>. Cantor is said to have returned the compliment by calling Kronecker <*der kleine Despot*>. I would like to draw the attention to facts of a more mathematical import.

Cantor was a student of Kronecker in Berlin and got interested in number theory. It may seen paradoxical that Cantor's well-known result on the canonical representation of a real-variable function by a trigonometric series is the birth certificate of the theory of sets (of derived points) and that the same theorem was to some extent arithmetized by a suggestion from Kronecker, as acknowledged twice by Cantor. The suggestion consisted in replacing a real-valued argument x by two arithmetical expressions $y+x$ and $y-x$ where y is a constant in order to cancel the coefficients $lim\, c_n = 0$ in the formula

$$lim\left(c_n sin\ nx\right) = 0 \qquad \text{for } n = \infty.$$

This correction is simple enough, but there is another topic that Kronecker did not live to evaluate, the normal form theorem which Cantor formulated in 1895 (four years after Kronecker's death).

1.1. The normal form theorem

The theorem has to do with a canonical representation of ordinals of the second class of numbers

$$\omega,\ \omega+1,\ \omega\cdot 2,\ldots,\ \omega^2,\ldots,\ \omega^\omega,\ldots,\ \omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega}}}}},\ \varepsilon_0,\ \varepsilon_\nu,\ldots,\ \varepsilon_\omega,\ \varepsilon_{\omega+1}$$

the cardinality of which is $2^{\aleph_0}$ where numbers ε are the critical numbers of the normal function $f(\zeta)=\omega^{\zeta}$. The set of recursive ordinals exhausts from above the arithmetical or polynomial operations by the operation of successor and limit constructions on the ordinals — ω_1 is the first ordinal not accessible from below.

The normal form is an ordinal polynomial

$$\zeta=\gamma_0\beta^{\alpha_n}+\gamma_1\beta^{\alpha_{n-1}}+\ldots+\gamma_{n-1}\beta^{\alpha}+\gamma_n$$

which is a finite sum (or series) $\sum\gamma_i\beta^i$ of decreasing powers of base β and integer coefficients $\gamma_i\geq 0$; it is therefore a rational function which stops by finite descent of its decreasing powers. The descent is here really a transfinite descent, that is transfinite induction for infinite powers α with a limit in the second number class

$$\lim_{n\to\omega}\left.\omega^{\omega^{\cdot^{\cdot^{\omega}}}}\right\}^{n}=\varepsilon_0$$

Cantor saw this normal form as an analogy with the finite case. Hessenberg has besides defined *natural* sums (and products) for the infinite case mimicking the finite situation. It is easy to translate the *ordinal* polynomial into an *ordinary* polynomial

$$P=b_0x^n+b_1x^{n-1}+\ldots+b_{n-1}x+b_n$$

where the x's represent Kronecker's indeterminates.

1.2. Transfinite induction

The ordinals of the normal form are generated by transfinite induction or complete induction on transfinite ordinals

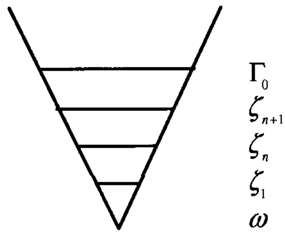
$$\forall\sigma\,[\forall t<\sigma\rightarrow A(t,x)\rightarrow A(\sigma,x)]\rightarrow\forall\sigma\,A(\sigma,x)$$

where t and σ are ordinals. Transfinite induction was introduced by Gentzen in proof theory and it plays a major rôle in contemporary set theory and proof theory.

The reflection principle in axiomatic set theory is written (for any limit ordinal $\beta>\alpha$)

$$\forall\sigma\,\exists\beta\,\forall x_1\in V_\beta\forall x_n\in V_\beta\{\phi\leftrightarrow\phi^{V_\beta}\}$$

— the limit ordinal is located in the cumulative rank structure V. The principle which is equivalent to the axiom of remplacement for an infinite rank V generates stronger and stronger axioms of infinity by climbing the ladder of ranks. In proof theory, the predicative ladder stops at ε_0

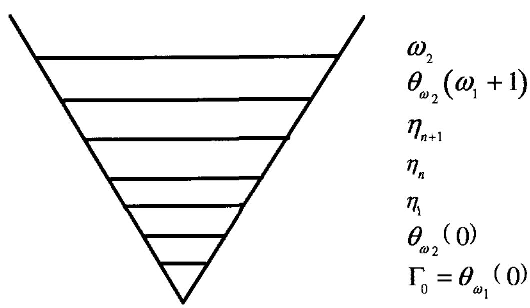


We have $lim_{n\to\infty} = \varepsilon_0$. The functions ζ are defined for the critical numbers of Veblen hierarchy

$$\varphi_0 = L\zeta \cdot \omega^{\zeta}$$
$$cr(\alpha):\{\zeta \mid \varphi_\alpha(\zeta) = \zeta\}.$$

Γ_0 is accessible from below and Cantor's normal form for the ordinals of the second class allows to climb back the cumulative hierarchy with the help of the ordinal polynomial of decreasing powers we have seen above.

Beyond the predicative universe, one can conceive a virtual universe, following Pohlers (1996)



The first non-denumerable ordinal ω_1 does not permit to climb any further step in the tansfinite ladder, but as a limit ordinal Γ_0 it can reflect the ω_1 hierarchy of ordinal η : here the collapse of the hierarchy is representable not by the degree of a polynomial, but by its height, that is the maximal length of the components or ordinal terms of a polynomial form

$$\omega_1 + \omega_2 + \ldots + \omega_n.$$

This polynomial diffraction can be expressed by a polynomial in the form

$$P = \left(b_0 x^n\right)_n + \left(b_1 x^{n-1}\right)_n + \ldots + \left(b_{n-1} x\right)_n + \left(b_n\right)_n$$

$$\vdots$$

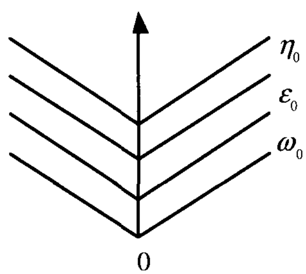
$$P = b_0 x + b_1 x + \ldots + b_{n-1} + b_n$$

where the x's are indeterminates in Kronecker's sense. I call this arithmetic hierarchy the absolute or canonical scale which I write

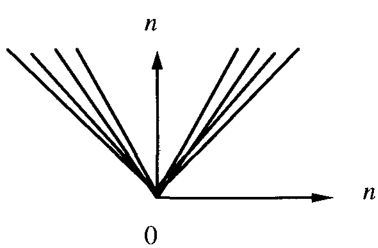
$$f\binom{n}{n} = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$$

where f is a polynomial of degree n and height n with integer coefficients a_i and indeterminates x. The arithmetical universe is thus the standard model of the first class, of the second class and of all the classes ω_n with $n < \omega$.

The polynomial diffraction alluded to above has a direct effect on the cumulative rank structure which can be represented as



for reflection by transfinite induction. Diffraction by infinite descent on the other side can be represented as



meaning that all extensions of the absolute scale n can be seen as associated scales of a diffractive fan, maybe reminescent of Brouwer's fan law for choice sequences. The reduction of transfinite ordinals to indeterminates ramifications of infinite descent is thus an illustration that transfinite induction is not equivalent to infinite descent from a constructive point of view — since it would require a double negation on an infinite set, as we have seen above (Chapter 4); but the two are indeed equivalent from a classical standpoint, if one allows for a realist account of infinite reflexive orders in the cumulative rank structure. To the constructivist, the higher ordinals are but the diffracted images of the natural numbers without further existential import than are indeterminates with a virtual reality only up to arithmetical existence.

1.3. Transfinite arithmetic versus general arithmetic

Kronecker's general arithmetic is a calculus of indeterminates associated to an arithmetic of integers and Kronecker's programme consists essentially in a reduction of algebraic quantities to polynomial arithmetic. No one ignores that an entire function — which takes all of its (finite) values — which is not a polynomial is *ipso facto* a transcendental function, that is non-algebraic. Kronecker's idea is to reach an arithmetical theory in which even transcendental functions are to be integrated to a finite calculus via the adjunction of indeterminates. The motto attributed to him "God has created the integers, the rest is the work of man", is not totally true, since on occasion Kronecker states that integers are constructions of the human mind (see Kronecker, 1968, II, 252-274).

For Kronecker, the arithmetical theory of algebraic quantities in built upon the theory of entire functions with integer coefficients <*ganze ganzzahlige algebraische Funktionen*> which included the theory of entire rational functions, that is the theory of forms or polynomials of a domain of rationality. Despite Cantor's hope of creating a transfinite arithmetic on the model of a pure finite arithmetic, the admission of limit-ordinals and critical numbers transforms arithmetic into analysis.

In his paper <*On the concept of number*>, (« Ueber den Zahlbegriff »), Kronecker shows how the real roots of an algebraic equation can be defined from a homogeneous polynomial with two indeterminates y and z

$$a_0y^n + a_1y^{n-1}z + a_2y^{n-2}z^2 + \ldots + a_nz^n$$

in the expression

$$f(z/y) = 1/y^nF(y,z)$$

by substituting integers to the indeterminates y and z — Kronecker even speaks of functions of indeterminate integers for integer-valued functions $F(y,z)$. As is well-known, transcendental numbers, *i.e.* which do not satisfy a polynomial with integer coefficients make their appearence in Liouville's definition (1884) of an irrational number X by the inequality

$$|X - p/q| < 1/q^n \qquad \forall n$$

where p/q is a rational number and $q > 1$. Kronecker sees this irrational number as indeterminate, but Cantor shows in 1874 that there is a bijection between algebraic numbers and integers via a polynomial with integer coefficients