PREFACE

"Internal" in the title is my translation for the German <*inhaltlich*> which one encounters frequently in the writings of Kronecker, Hilbert, Weyl or Brouwer. It has been rendered most of the time by "contentual" in English. Internal logic is the logic of content and it refers to what Hilbert called <*inhaltliches logisches Schliessen*> in his seminal paper « On the Infinite », but the terminology is frequent in various contexts. I have used the term in my 1994 paper « Hilbert and the internal logic of mathematics »[1] and I have exploited the idea in three books[2] published in France; however, the first use of <*logique interne*> in connection with Hilbert seems to have been introduced by the then young André Weil in his translation[3] of Hilbert's paper « Über das Unendliche » (1926). The more general meaning of "internal" logic can of course be found in the writings of many philosophers and logicians, from the French philosopher Léon Brunschvicg — who was influential on Poincaré at the turn of the century — to the contemporary American philosophers Carnap and Putnam among others, but my special usage is native.

Hilbert is the point of departure, but quickly I make a step backwards <*ein Schritt zurück*> and come to Kronecker. The book might be seen as a vindication of Kronecker's programme of a general arithmetic <*allgemeine Arithmetik*>, which I call polynomial arithmetic. The arithmetic of polynomials (or forms) is the heart of the matter and reaches out beyond Kronecker and Hilbert to the contemporary situation in the foundations of mathematics. In a forward step, I propose a proof of the self-consistency of arithmetic with infinite descent (chapter 4). The Fermat-Kronecker arithmetic (*FK* arithmetic for short) stands in sharp contrast to Dedekind-Peano arithmetic : it is not set-theoretic and does not employ Peano's induction postulate but Fermat's principle of infinite descent which is not equivalent to the principle of infinite induction from a constructivist point of view, to say the least. The distinction between internal and external consistency is duly examined in chapter 3, where I set the stage for the consistency proof.

The vindication of Kronecker's arithmetical foundations does not go without a critique of Hilbert's programme — begun in chapter 2 and continued in chapter 3 — on Kroneckerian grounds, since the late Hilbert has confessed in 1930 that his foundational stance <*finite Einstellung*> was quite close to Kronecker's finitism. I tend to relax the strictures of finitism by allowing "effinite" (from ex-finite) sequences in the sense of Brouwer's "infinitely proceeding sequences" and the constructive logic

---

[1] *Synthese* , 101 (1994), 1-14.

[2] *De la logique interne* (1991), *La logique interne des théories physiques* (1992), *Logique interne* (1997).

[3] « Sur l'infini », *Acta Mathematica*, 48 (1926), 91-122.

I advocate is more attuned to the theory of forms or homogeneous polynomials than to intuitionistic choice sequences. My critique of Hilbert is however less radical than my disaffection for Cantorian set theory, partly inspired by Kronecker's deep-seated reticence. My contention is that Hilbert's programme can be rescued if it is modified according to the canon of polynomial arithmetic, a close approximation of which is attempted in *Die Grundlagen der Mathematik*. But the finality of Kronecker's *Grundzüge einer arithmetischen Theorie der algebraischen Grössen* for the foundations of the arithmetical theory of polynomial algebra — the theory of forms — could not be reached and Gödel's incompleteness results reflect the failings of a finitary approach to set-theoretic arithmetic as Hilbert defined it and in some (obscure) way saw it to escape the finitist scope. In retrospect, Gödel accomplished negatively and paradoxically Hilbert's programme for the consistency problem, although he admitted that his result on consistency proofs did not contradict Hilbert's standpoint since it is possible that some kind of finitist proof cannot be represented in the formal system of Peano arithmetic and extensions thereof. Such a proof is described in chapter 4. Classically, the consistency problem is settled by Gentzen's and Ackermann's proof with the help of transfinite induction, which in spite of the diagonalization over an infinite set of natural numbers, reinstates Cantor's normal form theorem for transfinite ordinals up to ε_0, that is a disguised polynomial for indeterminate integers, as Kronecker undoubtedly would have termed it. In polynomial arithmetic, Cantor's diagonal is replaced by what can be called Cauchy's diagonal, *i.e.* the convolution product for series.

The debate between Cantor, Frege and Kronecker (chapter 5) could take place only posthumously, even if their posterity is unwilling to revive it. Kronecker is generally unacknowledged, except by expert mathematicians and historians (Weil and Edwards) or occasionally by constructivist descendents who are sometimes oblivious of their origins — Brouwer, Poincaré and the French semi-intuitionists, Weyl, Skolem, the Russian constructivist school and, nowadays, workers in constructive algebra or analysis when they are still active believers.

The revival of the arithmetization programme is witnessed by Nelson's rigorous reconstruction of predicative arithmetic — not to be confused with the Weyl-Feferman predicative programme — and to some extent by the interest in fragments of Peano arithmetic and bounded arithmetic which constitute in all a manifesto for a renewed proof-theoretic (and model-theoretic) investigation of arithmetic outside the traditional investigations on set-theoretic arithmetic, axiomatic set theory included. Complexity theory and in general the study of generative algorithms in theoretical computer science cannot but provide arithmetic with a rejuvenation of problems akin to the initial ideal of an overall arithmetization of mathematics; the topic is addressed in the final chapter of the book. Arithmetic geometry has in a likewise manner taken over algebraic geometry by returning to Kronecker's *Jugendtraum* in which analysis and algebra were striving towards an arithmetic interface (or intersection!). But without going into the polynomial dream of a unified physical theory (Witten style) — the polynomial invariants and dualities — one can still hope that the arithmetic theory of motives in algebraic theory (Grothendieck style) endows Kronecker's program with a sense of the actual. The French mathematician J. Dieudonné has pointed out that Grothendieck's notion of scheme (for algebraic varieties) has originated in

Kronecker's theory of modular systems — which is the ancestor of the contemporary theory of modular forms and elliptic curves.

In a different setting, contemporary quantum mechanics could be seen as an extension, not necessarily conservative, of Hilbert's program, insofar as Hilbert saw (with the assistance of von Neumann in that case) foundations of physics as the realization of an analytical apparatus *<analytischer Apparat>* congenial in more ways than one to an arithmetic theory of measurement (chapter 6).

I resist though the philosophical ambition of a "polynomy", that is the polynomialization or arithmetization of everything; the initial chapter tells us the story of the concepts of "indefinite" and "indeterminate" only to end up in the "effinite", which is neither definite nor indefinite. Perhaps, only the adoption of a minimal constructive (arithmetical) logic, a stringent internal logic, could act as a buffer-stop for that special train of thoughts.

In my reconstitution of the history of foundations of mathematics from Kronecker to Hilbert, the historical material consists in the published works of Kronecker and Hilbert. For Kronecker, we know that a great quantity of manuscripts (mainly lecture notes) has been lost in the Second World War and for Hilbert, I have quoted only publically available passages of the unpublished archives. I have no reason to believe that unpublished material, either from Kronecker or Hilbert, would affect to any degree my conceptual, to some extent ahistorical, reconstruction. In any case, the focus on Kronecker's major work *Grundzüge einer arithmetischen Theorie der algebraischen Grössen* is resolutely innovative in foundational studies, insofar as philosophers and logicians are totally ignorant of Kronecker and only a few mathematicians (*e.g.* H. Weyl and A. Weil) and historians of mathematics (*e.g.* H. Edwards) have paid due respect (or attention) to the great arithmetician. The philosopher or logician who wants to understand Hilbert and his references to Kronecker has better go back to Kronecker — and this is my sole historical contention — for a whole lot of Hilbert's mathematical and logical (and philosophical) ideas cannot be appreciated without the recognition of his intellectual debt to Kronecker. The claim to historical accuracy is thus overshadowed by the pioneering effort in the reviving (and revising) of Kronecker's program.

The ideas developed here have matured over a number of years and publications. The concepts of local negation, "effinite" quantifier, arithmetical logic and the emphasis on Fermat's descent were present before my encounter with Kronecker's general arithmetic of forms (polynomials); it gave the final impetus for the proof of the self-consistency of arithmetic which has been published only recently (in *Modern Logic*, 2000). The main theme of this book is announced in my 1994 paper in *Synthese*, already mentionned above. Other papers referred to have appeared in *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, *Archiv für mathematische Logik und Grundlagenforschung*, *Notre Dame Journal of Formal Logic*, *Dialectica*, *Philosophy of Science* and *International Journal of Theoretical Physics* among others. I have tried to integrate the recovered parts into a unified whole of historical, philosophical, logical and mathematical questions delimited by the foundational enterprise; in the process, I have not avoided cross-checks, sometimes plain repetitions, for the sake of an argument (and a style) which has hopefully gained in clarity, if not in simplicity.

Many people have contributed, directly or indirectly, in conversation or correspondence, in challenge or encouragement, to the views expressed here; over the years, R. Thom, H. Margenau, I. E. Segal, A. Wheeler, E. P. Wigner, G. Chew, H. Putnam, R. Gandy, A. N. Shanin, A. S. Troelstra, B. van Fraassen, A. Joyal, G. Reyes, more recently André Weil, E. Nelson, H. M. Edwards and D. van Dalen. E. Nelson's *Predicative Arithmetic* provided a timely incentive for the self-consistency proof, H. Edwards' appraisal of Kronecker's work has reinforced my commitment to Kronecker's general arithmetic and the late André Weil has encouraged me at an early stage in the exploration of Fermat's infinite descent. I have a special debt to Georg Kreisel who has aroused my critical instinct in foundational research from the start. Mathieu Marion has been helpful in providing collateral information. Finally, Frédérick Tremblay must be thanked for his vigilance and diligence in the electronic preparation of the manuscript. Financial help from the councils SSHRC of Canada and FCAR of Quebec during the period of conception is gratefully acknowledged.