Preface

Homology 3–spheres are closed oriented 3–manifolds having the homology of the 3–sphere. These objects may look rather special but they have played a prominent role in manifold topology for a long time. This book is a survey of various ideas and constructions used in their study, from the classical Rokhlin invariant through Casson's theory and its numerous generalizations up to the most recent gauge theoretical invariants. It is hardly possible to account for all the ramifications of this theory in one book. While trying to give as complete an account as possible, I was largely guided by my personal research interests in deciding which topics to cover in more detail.

Some results in the book have never appeared in monograph form before while others are rather well documented in the literature. The state of affairs in low dimensional topology by the early 1980s is described in Mandelbaum's survey [203]. The book [275] can be recommended as a gentle introduction to the Casson invariant aimed at graduate students. Akbulut and McCarthy's book [5] is a more advanced text on the topic. Walker's and Lescop's extensions of the Casson invariant are described in their respective books [302] and [182]. I highly recommend the surveys [139] and [164] for the gauge theoretical approach to the Casson invariant. The book [70] is a standard reference for Floer homology.

While working on this book, I greatly benefitted from conversations with Jim Bryan, Christopher Herald and Sławomir Kwasik. During the course of my work I held visiting positions in the Mathematics Departments at Duke, Princeton and Tulane Universities, and at Max Planck Institut für Mathematik in Bonn. I am thankful to these institutions for their hospitality and support. This work was also partially supported by NSF Grant DMS-0196523.

Coral Gables, Florida — *Nikolai Saveliev*
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5 Casson Invariant and Gauge Theory

In this chapter, we give an account of $SU(2)$–gauge theory in dimension three. We discuss C. Taubes' gauge-theoretical definition of the Casson invariant as (roughly) the Euler number of the gradient field of the Chern–Simons function. The Chern–Simons function plays a central role in modern understanding of homology 3–spheres, so we discuss it in some detail. An infinite dimensional analogue of Morse theory applied to the Chern-Simons function produces the instanton Floer homology which will be discussed in the next chapter. This gauge-theoretical approach to the Casson invariant leads to several extensions in a direction different from that of Walker and Lescop. One of the extensions we discuss is the $SU(3)$ Casson invariant of H. Boden and C. Herald. Another one is the Casson-type invariant for knots in integral homology spheres introduced by X.-S. Lin and C. Herald, and finally, the equivariant Casson invariant of integral homology spheres with a finite cyclic group action by O. Collin and the author.

5.1 Gauge Theory in Dimension 3

Let M be a closed oriented 3–manifold and $E \to M$ a principal $SU(2)$–bundle over M. By topological reasons, E is necessarily trivial, and we will fix a trivialization $E = M \times SU(2)$. The (affine) space of $SU(2)$–connections on E will be denoted by $\mathcal{A}$. The above trivialization identifies $\mathcal{A}$ with $\Omega^1(M, \mathfrak{su}(2))$, the linear space of differential 1–forms with matrix $\mathfrak{su}(2)$–coefficients.

A connection $A \in \mathcal{A}$ allows to lift any curve $\gamma : [0,1] \to M$ to a horizontal curve $\tilde{\gamma} : [0,1] \to E$ which is uniquely determined by a lift $\tilde{\gamma}(0)$ of $\gamma(0)$. If $\gamma(0) = \gamma(1)$ then $\tilde{\gamma}(0)$ and $\tilde{\gamma}(1)$ differ by an element of $SU(2)$, which is called the *holonomy* of A along γ and is denoted by $\operatorname{hol}_A(\gamma)$. Holonomy defines a map from the monoid of based loops in M to $SU(2)$.

The group of automorphisms of E is isomorphic to $\mathcal{G} = \operatorname{Map}(M, SU(2))$ and is called a *gauge group*. Its elements are called *gauge transformations*. They act on $\mathcal{A}$ by the rule $g^*A = g^{-1}dg + g^{-1}Ag$, where $g \in \mathcal{G}$ and $A \in \mathcal{A}$. The action of $\mathcal{G}$ on $\mathcal{A}$ is not free. The stabilizer of a connection $A \in \mathcal{A}$ equals the stabilizer of its holonomy group in $SU(2)$. A connection is called *irreducible* if its stabilizer is $\{\pm 1\}$, and it is called *reducible* otherwise. More specifically, connections with stabilizer $U(1)$ are called *abelian*, and those with stabilizer

$SU(2)$ are called *central*. The latter include the product connection θ, which is also called the *trivial* connection.

We let $\mathcal{B} = \mathcal{A}/\mathcal{G}$ and denote by $\mathcal{B}^*$ the subset of $\mathcal{B}$ consisting of $\mathcal{G}$–orbits of irreducible connections. Completed in appropriate Sobolev norms, $\mathcal{B}^*$ is a smooth infinite dimensional Banach manifold. The group $\mathcal{G}$ is not connected. In fact, $\pi_0\, \mathcal{G} = [M, SU(2)] = \mathbb{Z}$, the latter isomorphism given by the degree of $g : M \to SU(2)$. The homotopy exact sequence now implies that $\pi_1 \mathcal{B}^* = \mathbb{Z}$.

Associated with each connection $A \in \mathcal{A}$ is its *curvature*, $F_A = dA + A \wedge A$, which is an $\mathfrak{su}(2)$–valued differential 2–form on M (the wedge here stands for the matrix product on $\mathfrak{su}(2)$–coefficients and the wedge product on differential forms). A connection A is called *flat* if $F_A = 0$. The holonomy map for a flat connection A defines a homomorphism $A : \pi_1 M \to SU(2)$. This correspondence establishes a well-known identification

$$\operatorname{Hom}(\pi_1 M, SU(2))/SO(3) = \{\, \text{Flat } SU(2) \text{ connections on } M\,\}/\mathcal{G}.$$

Irreducible representations on the left correspond to irreducible flat connections on the right, so that the representation space $\mathcal{R}^*(M)$ can be viewed as a subset of $\mathcal{B}^*$. Abelian (resp. central) connections correspond to abelian (resp. central) representations, see Section 3.2.1.

A choice of Riemann metric on M provides us with an L^2–inner product on the spaces $\Omega^p(M)$ of differential p–forms for each $p \geq 0$. Together with the positive-definite pairing $(a, b) \mapsto -\operatorname{tr}(ab)$ on $\mathfrak{su}(2)$ it gives an L^2–inner product $(\ ,\ )$ on $\Omega^p(M, \mathfrak{su}(2))$. Given orientation on M, this defines the *Hodge star operator* $* : \Omega^p(M, \mathfrak{su}(2)) \to \Omega^{3-p}(M, \mathfrak{su}(2))$ by the rule

$$-\int_M \operatorname{tr}(a \wedge *b) = (a, b).$$

The tangent space to $\mathcal{A}$ at a point $A \in \mathcal{A}$ is of course $\Omega^1(M, \mathfrak{su}(2))$, and the normal space to the $\mathcal{G}$–orbit of A may be identified with the kernel of the differential operator

$$d_A^* = - * d_A * \,:\, \Omega^1(M, \mathfrak{su}(2)) \to \Omega^0(M, \mathfrak{su}(2))$$

which is formally adjoint to the operator d_A. Here, $d_A u = du + [A, u]\, (= du + Au - uA)$ is the covariant derivative of u with respect to A.

The operator d_A extends by the Leibniz rule to all spaces $\Omega^p(M, \mathfrak{su}(2))$. If A is flat, we obtain an elliptic complex

$$\Omega^0(M, \mathfrak{su}(2)) \xrightarrow{d_A} \Omega^1(M, \mathfrak{su}(2)) \xrightarrow{d_A} \Omega^2(M, \mathfrak{su}(2)) \xrightarrow{d_A} \Omega^3(M, \mathfrak{su}(2))$$

whose cohomology $H^*(M, \operatorname{ad} A)$ is the cohomology of M with coefficients in the twisted flat bundle $E \times_{SU(2)} \mathfrak{su}(2)$. In particular, $H^0(M, \operatorname{ad} A)$ can be identified with the Lie algebra of the stabilizer of A, and $H^1(M, \operatorname{ad} A)$ with the group cohomology of $\pi_1 M$ with coefficients in representation (3.5).

5.2 Chern-Simons Function

Let M be a closed oriented 3-dimensional manifold. For any connection $\alpha \in \mathcal{A}$, we define the *Chern-Simons function* of α by the formula

$$\mathbf{cs}(\alpha) = \frac{1}{8\pi^2} \int_M \operatorname{tr}(\alpha \wedge d\alpha + \frac{2}{3}\, \alpha \wedge \alpha \wedge \alpha). \tag{5.1}$$

The Stokes' Theorem implies that, for any smooth compact oriented 4-manifold X with $\partial X = M$ and an $SU(2)$–connection A on X which restricts to α over M,

$$\mathbf{cs}(\alpha) = \frac{1}{8\pi^2} \int_X \operatorname{tr}(F_A \wedge F_A). \tag{5.2}$$

According to Chern-Weyl theory, the latter integral over a closed manifold represents a second Chern number and, in particular, is an integer. This implies that $\mathbf{cs}(g^*\alpha) = \mathbf{cs}(\alpha) + \deg(g)$, where $g \in \mathcal{G}$ so that Chern-Simons function is well-defined as a map $\mathbf{cs} : \mathcal{B} \to \mathbb{R}/\mathbb{Z}$.

Another equivalent definition of the Chern-Simons function is as follows. Given a connection α over M, we can take a path $[0,1] \to \mathcal{A}$ from the trivial connection to α in the trivial $SU(2)$–bundle. This path determines a connection A in the trivial $SU(2)$-bundle over $[0,1] \times M$, and we let

$$\mathbf{cs}(\alpha) = \frac{1}{8\pi^2} \int_{[0,1]\times M} \operatorname{tr}(F_A \wedge F_A). \tag{5.3}$$

The set of values of **cs** on its critical points is an invariant of a 3–manifold called the *Chern–Simons invariant* Methods for computing this invariant and examples of computations can be found in Auckly [18], Fintushel-Stern [83] and Kirk-Klassen [165]. In all known cases, the values of **cs** on its critical points are rational numbers. Whether the same is true in general is an open problem.

Example 5.1. The Chern–Simons invariant for a lens space $L(p,q)$ is given by $-n^2 r/p$ for $n = 0, \ldots, [p/2]$ where r is any integer satisfying $qr = -1 \bmod p$, see [165]. Note that the Chern–Simons invariant distinguishes homotopy inequivalent lens spaces, compare with Section 1.2.1, but it cannot distinguish homotopy equivalent non-homeomorphic lens spaces.

Example 5.2. The Chern-Simons invariants for Seifert fibered homology spheres $\Sigma(a_1, \ldots, a_n)$ are calculated in [83] and [165]. Suppose that $\Sigma(a_1, \ldots, a_n)$ has Seifert invariants b, $(a_1, b_1), \ldots, (a_n, b_n)$ with b even, see Section 1.1.4. Moreover, if one of the a_i is even, assume it is a_1 and arrange the Seifert invariants so that the b_i with $i \neq 1$ are even. If $\alpha : \pi_1 \Sigma(a_1, \ldots, a_n) \to SU(2)$ is a representation with rotation numbers $(\ell_1, \ldots, \ell_n)$, see Section 3.5.2, then

$$\mathbf{cs}(\alpha) = -\frac{e^2}{4a_1 \cdots a_n} \bmod \mathbb{Z} \quad \text{where} \quad e = a_1 \cdots a_n \cdot \sum_{i=1}^{n} \frac{\ell_i}{a_i}. \tag{5.4}$$

Example 5.3. The results of Examples 5.1 and 5.2 are instances of the following more general theorem, see [165]. Let $k \subset M$ be a knot in a closed oriented 3–manifold M with exterior K, and let curves m and ℓ form a basis in $\pi_1(\partial K) = \mathbb{Z}^2$. Given two representations $\alpha_0, \alpha_1 : \pi_1 M \to SU(2)$, suppose that there is a path $\alpha_t : \pi_1 K \to SU(2)$, $0 \le t \le 1$, from α_0 to α_1 (both restricted to $\pi_1 K$). Let $(\mu(t), \lambda(t))$ be a path in $\mathbb{R}^2$ such that

$$\alpha_t(m) = e^{2\pi i \mu(t)}, \quad \alpha_t(\ell) = e^{2\pi i \lambda(t)}.$$

Then

$$\mathbf{cs}(\alpha_1) - \mathbf{cs}(\alpha_0) = -2 \int_0^1 \lambda(t)\mu'(t)\, dt \in \mathbb{R}/\mathbb{Z}. \tag{5.5}$$

This means that if we know the image of the path $\alpha_t : \pi_1 K \to SU(2)$, $0 \le t \le 1$, in the pillowcase, see Section 3.3.1, then we can compute the difference $\mathbf{cs}(\alpha_1) - \mathbf{cs}(\alpha_0)$. Thus the difference is determined by the pillowcase image of $\mathcal{R}(K)$. Application of this result to surgeries on the unknot gives results of Example 5.1, and its application to the surgeries on a regular fiber in a Seifert fibered homology sphere gives results of Example 5.2.

This method does not work in the situations when the representation variety $\mathcal{R}(K)$ is not connected. This happens for instance for the figure-eight knot, see Figure 3.3. This difficulty can sometimes be overcome by passing to the complex character variety $\mathrm{Hom}(\pi_1 K, SL(2,\mathbb{C}))/SL(2,\mathbb{C})$. In particular, the Chern–Simons invariants of surgeries on the figure-eight knot can be computed by this method, see [165].

An affirmative answer to the following question would make this theory more useful. Given a closed manifold M and representations $\alpha_0, \alpha_1 : \pi_1 M \to SU(2)$, does there exist a knot (or link) $k \subset M$ so that the restrictions of α_0 and α_1 onto the exterior K of k lie on a path component of $\mathcal{R}(K)$?

Example 5.4. The formula (5.5) was used to compute the Chern-Simons invariants for torus bundles over a circle in [165] and for higher genus surface bundles in [15]. The formula ((5.5) was generalized to include general decompositions of 3–manifolds along tori in [166] and [170].

Example 5.5. Two representations, $\alpha_M : \pi_1 M \to SU(2)$ and $\alpha_N : \pi_1 N \to SU(2)$ are called *flat cobordant* if there exists an oriented cobordism W between M and N and a representation $\alpha : \pi_1 W \to SU(2)$ restricting to α_M and α_N over M and N, respectively. It follows immediately from (5.2) that $\mathbf{cs}(\alpha_M) = \mathbf{cs}(\alpha_N)$. Using this observation and the formula (5.5) David Auckly [18] developed a method which allows to compute the Chern–Simons invariants for all graph manifolds and for some hyperbolic manifolds.

Remark 5.6. The definition of $\mathbf{cs}(A) \in \mathbb{R}/\mathbb{Z}$ for $SU(2)$–connections A can be easily adapted to define $\mathbf{cs}(A') \in \mathbb{R}/4\mathbb{Z}$ for $SO(3)$–connections A'. The two Chern-Simons functions are related by the formula $4 \cdot \mathbf{cs}(A) = \mathbf{cs}(\mathrm{ad}_A)$ where ad_A is the induced $SO(3)$–connection in the adjoint bundle. If A is a flat connection viewed as an $SU(2)$ representation then the representation corresponding to ad_A is obtained by composing A with the adjoint representation $\mathrm{ad} : SU(2) \to SO(3)$.

The Chern-Simons function was first defined as a secondary characteristic class in [54]. It was shown later in [12], [13] and [14] that the ρ_α-invariants, which are defined as real numbers, are congruent to $\mathbf{cs}(\alpha)$ modulo $\mathbb{Z}$. Thus, $\mathbf{cs}(\alpha)$ can be viewed as the non-integral part of ρ_α, see Remark 6.17.

5.3 The Casson Invariant from Gauge Theory

A gauge–theoretical definition of the Casson invariant was given by Taubes [292]. His construction was motivated by an analogy with finite dimensional Morse theory, namely, the derivation of the Euler characteristic from critical points of a Morse function. Another description of the Casson invariant motivated by Seiberg–Witten gauge theory was recently suggested by Ozsváth and Szabó [254].

5.3.1 Morse Theory and Euler Characteristic

Let $h : X \to \mathbb{R}$ be a smooth function on a closed smooth manifold X. The *Hessian* of h at a point $p \in X$ is the bilinear form on the tangent space T_pX given in local coordinates by the matrix

$$\mathrm{Hess}_p\, h = \left(\frac{\partial^2 h}{\partial x_i \partial x_j}\right)(p). \tag{5.6}$$

If p is a critical point of h, we define its *Morse index* $\mu(p)$ as the number of negative eigenvalues of $\mathrm{Hess}_p(h)$ counted with multiplicities. We call $h : X \to \mathbb{R}$ a *Morse function* if its Hessian is non-degenerate at all of its critical points. The condition of being Morse ensures that the critical points of h are isolated and hence there are only finitely many of them. Then, according to the Poincaré–Hopf theorem,

$$e(X) = \sum_p (-1)^{\mu(p)},$$

where $e(X)$ is the Euler characteristic of X and the summation on the right extends over all critical points of h.

5.3.2 Critical Points of cs and Spectral Flow

Let Σ be an integral homology sphere endowed with a Riemannian metric and let $\mathbf{cs} : \mathcal{B}^* \to \mathbb{R}/\mathbb{Z}$ be the Chern–Simons function (5.1). The L^2*–gradient* of $\mathbf{cs}$ can be easily computed as

$$(\nabla\,\mathbf{cs})(A) = -\frac{1}{4\pi^2}\,*F_A, \tag{5.7}$$

which can be viewed as a vector field on the (infinite dimensional) manifold $\mathcal{B}^*$. The critical points of **cs** are (the gauge equivalence classes of) irreducible flat connections, which we identified earlier with the points in $\mathcal{R}^*(\Sigma)$.

The *Hessian* of **cs** at a connection A is the first order differential operator

$$\left(-\frac{1}{4\pi^2}\right)\,*\,d_A : \ker d_A^* \to \ker d_A^*, \tag{5.8}$$

which is essentially self–adjoint. Let us postpone discussion on the non-degeneracy of this operator until next section and see instead how it can be used to define indices of the critical points of **cs**. The problem is of course that the spectrum of (5.8) is unbounded in both directions, hence we cannot define the index of a critical point as the number of negative (or positive) eigenvalues, as we did it in the finite–dimensional Morse theory. One way around this problem is to compare the Hessians at two critical points by seeing how many eigenvalues change sign along a path connecting them. This will give a relative index, and the problem then will be to pick an overall normalization.

The natural choice of a flat connection with which to compare all other flat connections is the trivial connection θ. Unfortunately, θ does not belong to $\mathcal{B}^*$. Because of this, it is useful to replace the operator (5.8) by a Fredholm operator which gives the same relative index between irreducible flat connections but which also makes sense at reducibles. We define such an operator by

$$K_A = \begin{pmatrix} 0 & d_A^* \\ d_A & -*\,d_A \end{pmatrix} \;:\; (\Omega^0 \oplus \Omega^1)\,(M, \mathfrak{su}(2)) \to (\Omega^0 \oplus \Omega^1)\,(M, \mathfrak{su}(2)). \tag{5.9}$$

For any connection A, the operator K_A is an elliptic operator on M. Its L^2–completion is a self–adjoint Fredholm operator. It has pure point real spectrum without accumulation points, which is unbounded in both directions.

Let A_0 and A_1 be flat connections and consider a continuously differentiable path of operators $K_{A(t)}$ with $A(0) = A_0$ and $A(1) = A_1$. The one–parameter family of spectra of operators $K_{A(t)}$ can be viewed as a collection of spectral curves in the plane connecting the spectrum of K_{A_0} to the spectrum of K_{A_1} see Figure 5.1. These curves are continuously differentiable functions of t, at least near zero. Consider the straight line connecting the points $(0, -\delta)$ and $(1, \delta)$ where $\delta > 0$ is chosen smaller than the absolute value of any non–zero eigenvalue of K_{A_0} and K_{A_1}. The number of eigenvalues, counted with multiplicities, which cross from negative to positive side of this line minus the number which cross from positive to negative is well–defined and finite along a generic path. This number is called the *spectral flow* of the family along the path. Note that we choose a $\delta > 0$ to help us deal with flat connections A whose associated operator K_A has non-trivial kernel; away from such connections we could choose $\delta = 0$.

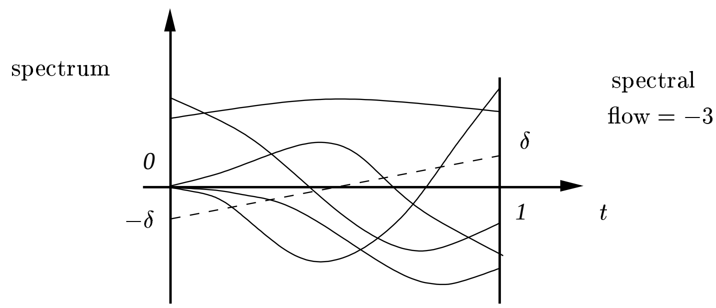

Fig. 5.1.

The spectral flow only depends on the homotopy class rel $\{0,1\}$ of the path $t \to A(t)$. Therefore, it defines a locally constant function on the space of continuous paths between A_0 and A_1. We denote this function by $\operatorname{sf}(A_0, A_1)$. Choosing different A_0 or A_1 within their gauge equivalence classes changes $\operatorname{sf}(A_0, A_1)$ to $\operatorname{sf}(A_0, A_1) + 8k$ for some integer k, see Corollary 6.3. Therefore, $\operatorname{sf}(A_0, A_1) \bmod 8$ is well defined on gauge equivalence classes of A_0 and A_1.

Applications of the spectral flow in this chapter will only use its modulo 2 reduction. The full strength of the modulo 8 spectral flow will be explored in Chapter 6, together with various spectral flow formulas.

5.3.3 Non–degenerate Case

Let Σ be an integral homology sphere such that $\mathcal{R}^*(\Sigma)$ is non-degenerate, that is, $H^1(\pi_1\Sigma, \operatorname{ad}\alpha) = 0$ for every representation $\alpha \in \mathcal{R}^*(\Sigma)$, see Lemma 3.16.

Lemma 5.7. *For every $\alpha : \pi_1\Sigma \to SU(2)$, the kernel of the operator K_α is isomorphic to $H^0(\pi_1\Sigma, \operatorname{ad}\alpha) \oplus H^1(\pi_1\Sigma, \operatorname{ad}\alpha)$.*

Proof. Since K_α is a self–adjoint operator and α is flat, we have $\ker K_\alpha = \ker K_\alpha^2$ where

$$K_\alpha^2 = \begin{pmatrix} d_\alpha^*\, d_\alpha & 0 \\ 0 & d_\alpha^*\, d_\alpha + d_\alpha\, d_\alpha^* \end{pmatrix}.$$

Therefore, the kernel of K_α is represented by harmonic 0– and 1–forms with coefficients in the flat bundle ad_α. Such forms are in one-to-one correspondence with vectors in $H^0(\pi_1\Sigma, \operatorname{ad}\alpha) \oplus H^1(\pi_1\Sigma, \operatorname{ad}\alpha)$, by Hodge theorem.

The cohomology groups $H^0(\pi_1 \Sigma, \operatorname{ad}\alpha)$ vanish for irreducible representations α, therefore, $\mathcal{R}^*(\Sigma)$ is non-degenerate if and only if all irreducible flat connections are non-degenerate as critical points of the Chern–Simons function.

Suppose that $\mathcal{R}^*(\Sigma)$ is non-degenerate, then it is finite. For any $\alpha \in \mathcal{R}^*(\Sigma)$ define its *Floer index* by the formula

$$\mu(\alpha) = \operatorname{sf}(\theta, \alpha) \bmod 8, \tag{5.10}$$

compare with Section 6.2.2. Taubes showed in [292] that the quantity

$$\frac{1}{2} \sum_{\alpha \in \mathcal{R}^*(\Sigma)} (-1)^{\mu(\alpha)} \tag{5.11}$$

is independent of the metric on Σ and equals (up to an overall sign) the Casson invariant of Σ, compare with Theorem 5.9. Thus in the non-degenerate case, both (3.9) and (5.11) count the same points in the space $\mathcal{R}^*(\Sigma)$, and the essence of Taubes' theorem is that the two counts are consistent, one coming from intersection theory and the other from gauge theory. A formula similar to (5.11) will hold for all integral homology spheres Σ after we take care in the next section of (possible) degeneracies in $\mathcal{R}^*(\Sigma)$.

5.3.4 Perturbations

Let Σ be an integral homology 3–sphere and let $\gamma_i : S^1 \times D^2 \to \Sigma$, $i = 1, \dots, n$, be a collection of embeddings of solid tori in Σ with disjoint images. We will think of the γ_i as loops in Σ and refer to $\{\gamma_i\}$ as a link. Let $\eta(x)$ be a smooth rotationally symmetric bump function on the unit disc D^2 with the support away from the boundary of D^2 and with integral one. Define an *admissible perturbation function* $h : \mathcal{B} \to \mathbb{R}$ by the formula

$$h(A) = \sum_{i=1}^{n} \int_{D^2} h_i(\operatorname{tr} \operatorname{hol}_A(\gamma_i(S^1 \times \{x\}))) \cdot \eta(x)\, d^2x, \tag{5.12}$$

where h_i, $i = 1, \dots, n$, are smooth functions (one averages over the tubular neighborhood by analytical reasons). The function h is gauge invariant because a gauge transformation only changes holonomy by conjugation. Given an admissible perturbation function h, the *perturbed flat moduli space*

$$\mathcal{R}_h(\Sigma) = \{\, A \mid *F_A - 4\pi^2\,(\nabla h)(A) = 0 \,\}/\mathcal{G},$$

where ∇h is the L^2–gradient of h, is the critical point set of $\mathbf{cs} + h : \mathcal{B} \to \mathbb{R}/\mathbb{Z}$. Extend the definition (5.9) to obtain the operator

$$K_{A,h} = \begin{pmatrix} 0 & d_A^* \\ d_A & -*\, d_A + 4\pi^2\, \operatorname{Hess}_A h \end{pmatrix} \tag{5.13}$$

where $\mathrm{Hess}_A\, h$ is the Hessian of h at A. A critical point A of the perturbed Chern-Simons function $\mathbf{cs} + h$ is *non-degenerate* if and only if $\ker K_{A,h} = 0$. We call $\mathcal{R}_h(\Sigma)$ *non-degenerate* if all of its points are non-degenerate. The following result is proved in [137] and [292], see also [30].

Theorem 5.8. *For any admissible perturbation function h, the space $\mathcal{R}_h(\Sigma)$ is compact. Moreover, there exists a link $\{\gamma_i\}$ with sufficiently many components such that, for a generic choice of the functions $\{h_i\}$, the critical point set $\mathcal{R}_h(\Sigma)$ is non-degenerate and contains the trivial connection.*

Proof. Uhlenbeck's compactness theorem [299] can be adapted to prove the compactness of $\mathcal{R}_h(\Sigma)$, see [138]. Recall that the compactness of the non-perturbed space $\mathcal{R}(\Sigma)$ is immediate from the fact that $\mathcal{R}(\Sigma)$ is a real algebraic variety, see Section 3.2.1. To prove the second part of the proposition, one needs to find an abundant collection of admissible perturbation functions such that, in any direction tangent to $\mathcal{R}(\Sigma)$, there is a function whose derivative is non-zero. Given such a collection, an argument with the implicit function theorem completes the proof.

Let h be a generic admissible perturbation function as in Theorem 5.8 so that $\mathcal{R}_h(\Sigma)$ is non-degenerate and hence finite. Denote by $\mathcal{R}^*_h(\Sigma)$ the subset of $\mathcal{R}_h(\Sigma)$ consisting of the orbits of irreducible perturbed flat connections. Given $A \in \mathcal{R}^*_h(\Sigma)$, define its *Floer index* $\mu(A)$ as the spectral flow mod 8 of the family of operators $K_{A(t),h}$ where $A(t)$ is a path of connections from θ to A.

Theorem 5.9. *(Taubes [292]) The quantity*

$$\frac{1}{2} \sum_{A \in \mathcal{R}^*_h(\Sigma)} (-1)^{\mu(A)} \tag{5.14}$$

is independent of h and the metric on Σ and equals up to an overall sign the Casson invariant of Σ.

Remark 5.10. This is a rather technical result and we refer the reader to Taubes' paper [292] for the proof. We only mention that the requirement that Σ be an integral homology sphere simplifies things a lot in that it prevents the existence of non-trivial reducible flat connections before and after a small perturbation. This allows one to use the standard cobordism argument to show that the quantity (5.14) is independent of the choice of perturbation h, since the parametrized moduli space

$$\bigcup_{t \in [0,1]} \mathcal{R}^*_{h(t)}(\Sigma) \times \{t\} \tag{5.15}$$

gives a cobordism between $\mathcal{R}^*_{h(0)}(\Sigma)$ and $\mathcal{R}^*_{h(1)}(\Sigma)$. If the assumption that $H_1(\Sigma, \mathbb{Z}) = 0$ is dropped, there might exist non-trivial reducible flat connections which will prevent (5.15) from being compact. In the case of rational homology spheres, this problem is addressed in preprint [187]. This leads to a gauge theoretical interpretation of the Walker invariant.

5.3.5 Morse–type Perturbations

In practice, it is common that a degenerate representation variety $\mathcal{R}^*(\Sigma)$ turns out to be non-degenerate in the following weaker sense: at every point A in $\mathcal{R}^*(\Sigma)$, the kernel of the Hessian is equal to $T_A\mathcal{R}^*(\Sigma)$ and the Hessian is non-degenerate in the direction of the normal bundle of $\mathcal{R}^*(\Sigma)$ in $\mathcal{B}^*$. In particular, $\mathcal{R}^*(\Sigma)$ is smooth. If this is the case we say that $\mathcal{R}^*(\Sigma)$ is non-degenerate in *the Morse–Bott sense*. Each connected component of the manifold $\mathcal{R}^*(\Sigma)$ admits a Morse function, and these functions combined together can be used instead of the admissible perturbations (5.12) to perturb $\mathcal{R}^*(\Sigma)$ into a finite collection of non-degenerate points. Of course, one needs to show that the invariants resulting from the two different sets of perturbations are the same. This is widely believed to be true although we are not aware of any proof of this fact that has ever appeared in the literature, at least in such a generality. The following is based on Christopher Herald's notes he kindly supplied us with.

Theorem 5.11. *Suppose that $\mathcal{R}^*(\Sigma)$ is non-degenerate in the Morse–Bott sense. Then there exists an admissible perturbation function $h : \mathcal{B}^* \to \mathbb{R}$ such that its restriction to each connected component $\mathcal{R}_k$ of $\mathcal{R}^*(\Sigma)$ is a Morse function $f_k : \mathcal{R}_k \to \mathbb{R}$. Moreover, for any sufficiently small $\varepsilon > 0$ there is an open neighborhood U_k of $\mathcal{R}_k$ disjoint from other components of $\mathcal{R}^*(\Sigma)$ such that all the critical points of $\mathbf{cs} + \varepsilon \cdot h$ in U_k are non-degenerate and can naturally be identified with the critical points of f_k.*

This theorem is proved in [29]. Its downside is that we do not have control of the Morse functions f_k, and they may turn out to be unnecessarily complicated. Before we go on, we need to slightly enlarge the class of admissible perturbation functions. Define $g_i : \mathcal{B}^* \to \mathbb{R}$ by the formula

$$g_i(A) = \int_{D^2} h_i(\operatorname{tr}\operatorname{hol}_A(\gamma_i(S^1 \times \{x\}))) \cdot \eta(x)\, d^2x,$$

and view an m–tuple $(g_1, \dots, g_m)$ as a function from $\mathcal{B}^*$ to $\mathbb{R}^m$. We extend the class of perturbations by allowing to compose $(g_1, \dots, g_m)$ with arbitrary smooth functions $\psi : \mathbb{R}^m \to \mathbb{R}$, not just linear functions $\psi(t_1, \dots, t_m) = t_1 + \dots + t_m$ as in (5.12). This extended class of admissible perturbations leads to the same invariants.

Theorem 5.12. *Let $\mathcal{R}^*(\Sigma)$ be non-degenerate in the Morse–Bott sense, and let $\mathcal{R}_1, \dots, \mathcal{R}_n$ be its connected components. For any choice of Morse functions $f_k : \mathcal{R}_k \to \mathbb{R}$, there is an admissible perturbation function h such that the restriction of h onto $\mathcal{R}_k$ is equal to f_k for each $k = 1, \dots, n$.*

Proof. We start with an abundant collection of admissible perturbation functions as in the proof of Theorem 5.8. We enlarge this collection if necessary so that there are admissible perturbation functions $g_1, \dots, g_m$ which separate points in the flat moduli space $\mathcal{R}(\Sigma)$. The proof that this is possible is essentially due to Taubes [292], and we only recast it briefly. Abundance guarantees that finitely many admissible perturbation functions are needed to separate points $(A, A') \in \mathcal{R}(\Sigma) \times \mathcal{R}(\Sigma)$ in an open neighborhood W of the diagonal. For any pair (A, A') not on the diagonal, there is a curve γ in Σ such that the trace of the holonomy along γ separates points in an open neighborhood of (A, A'). Compactness of $\mathcal{R}(\Sigma) \times \mathcal{R}(\Sigma) \setminus W$ allows us to choose a finite collection of additional curves to assure the point separation property.

The point separation property for $g_1, \dots, g_m$ is not nearly as strong as saying that any smooth function on the $\mathcal{R}_k$ can be obtained as a linear combination of $g_1, \dots, g_m$. This is the reason why we enlarged the class of admissible perturbations to include functions $\psi(g_1, \dots, g_m)$ for any smooth $\psi : \mathbb{R}^m \to \mathbb{R}$. The theorem now follows by composing the function $(g_1, \dots, g_m) : \mathcal{A} \to \mathbb{R}^m$ with the f_k and suitable bump functions on $\mathbb{R}^m$.

5.3.6 Casson Invariant and Seiberg–Witten Equations

Let Σ be an oriented integral homology sphere equipped with a Riemannian metric g and consider the bundle $\mathrm{Cl}(T\Sigma) \otimes \mathbb{C}$ of complexified Clifford algebras over Σ. There exists a unique $SU(2)$ vector bundle W_0 over Σ acted upon by $\mathrm{Cl}(T\Sigma) \otimes \mathbb{C}$ such that the action of the volume form of Σ is the identity on W_0. Let $W = W_0 \otimes L$ where L is a trivial complex line bundle over Σ, then W is a $U(2)$–bundle. The Levi–Civita connection on $T\Sigma$ lifts to a connection on W_0, and for any choice of a $U(1)$–connection on L we have the corresponding Dirac operator $\eth_A : \Gamma(W) \to \Gamma(W)$ on the sections of W.

Let $\mathcal{A} = \mathcal{C} \times \Gamma(W)$ where $\mathcal{C}$ is the space of smooth $U(1)$–connections in L. The gauge group $\mathcal{G} = \mathrm{Map}(\Sigma, S^1)$ acts on $\mathcal{A}$ by the rule $g^*(A, \psi) = (A + 2g^{-1}dg, g^{-1}\psi)$. This action is free on the subset $\mathcal{A}^*$ away from the points $(A, 0)$. In proper Sobolev completions, $\mathcal{B}^* = \mathcal{A}^*/\mathcal{G}$ is an infinite dimensional Banach manifold. Define the *Chern–Simons–Dirac function* $\mathbf{cs}_D : \mathcal{B}^* \to \mathbb{R}$ by the formula

$$\mathbf{cs}_D(A, \psi) = \int_\Sigma A \wedge F_A + \int_\Sigma \langle \psi, \eth_A \psi \rangle_{\mathrm{Re}} \, dV,$$

compare with (5.1). The set of critical points of $\mathbf{cs}_D$ is identified with the moduli space $\mathcal{M}$ of solutions of Seiberg–Witten equations, see [177] and [228]. After a generic perturbation, the moduli space $\mathcal{M}$ is finite and non–degenerate and its points are canonically oriented. A count of these points results in an invariant $\chi_{SW}(\Sigma, g) = \#\mathcal{M}$, which is independent of all arbitrary choices made in its definition but one : it depends on the Riemannian metric g. However, according to [53], the quantity

$$\alpha(\Sigma) = \chi_{SW}(\Sigma, g) - \left(\operatorname{ind} \eth + \frac{1}{8} \operatorname{sign} W\right)$$

is a topological invariant of Σ, where W is any compact spin 4–manifold with $\partial W = \Sigma$ and $\eth$ is the Dirac operator on W satisfying the boundary conditions of [12]. The invariant $\alpha(\Sigma)$ can also be expressed in terms of $\chi_{SW}(\Sigma, g)$ and certain η–invariants of Σ, see for example [198].

Theorem 5.13. *For any integral homology sphere Σ, the quantity $\alpha(\Sigma)$ is equal up to an overall sign to the Casson invariant $\lambda(\Sigma)$.*

This result was conjectured to be true by Kronheimer and Mrowka. It was first proved modulo 2 in [53] then checked by an explicit calculation for all Brieskorn homology spheres $\Sigma(p, q, r)$ in [248]. A complete proof of this conjecture can be found in [198] where it is shown that the invariant α satisfies the Casson's surgery formula and hence coincides with λ up to an overall sign by Theorem 3.1. The proof in [198] crucially relies on the paper of Meng and Taubes [213]. Another approach to proving Theorem 5.13 can be found in a series of preprints [205].

The above invariant arising from the Seiberg–Witten gauge theory bears close resemblance to the invariant defined by Ozsváth and Szabó [254] using Heegaard splittings and theta divisors on Riemann surfaces. Their invariant is defined for all rational homology spheres and, if properly normalized, coincides with the Walker invariant (and the Casson invariant for integral homology spheres).

5.4 Casson–type Invariants of Knots

By Casson–type invariants of knots we mean the invariants introduced by X.-S. Lin [199] and C. Herald [137]. They are obtained by (creatively) counting irreducible $SU(2)$ representations of the knot group which have a fixed trace along the knot meridian. It turns out that these invariants can be expressed in terms of the (regular) Casson invariant and certain equivariant knot signatures. Ramifications of these ideas lead to the equivariant Casson invariant, see Section 5.5, and the Floer homology of knots, see Section 6.7.5.

5.4.1 Representation Varieties of Knot Groups

Let $k \subset \Sigma$ be a knot in an oriented integral homology 3–sphere Σ and let K be its exterior. Then K is a compact oriented 3–manifold with boundary $\partial K = T^2$, a 2–dimensional torus. The inclusion $\partial K \to K$ induces a map of representation varieties $r : \mathcal{R}(K) \to \mathcal{R}(T^2)$, where $\mathcal{R}(T^2)$ is the pillowcase variety, see Section 3.3.1. In a generic situation, the variety $\mathcal{R}(K)$ and its image in $\mathcal{R}(T^2)$ are described by the following theorem of Herald [138].

Theorem 5.14. *For a generic admissible perturbation h, the perturbed character variety $\mathcal{R}_h(K)$ consists of two central orbits, a smooth open arc of abelian orbits with one non-compact end limiting to each central orbit, and a smooth 1-dimensional manifold of irreducible orbits with finitely many ends, each limiting to a different point on the abelian arc. Suppose that the abelian arc is parametrized by ϕ, $0 < \phi < \pi$. If a point on the abelian arc with coordinate ϕ is the limiting points of an irreducible orbit then $e^{2i\phi}$ is a root of the Alexander polynomial $\Delta_k(t)$.*

The map $r : \mathcal{R}_h(K) \to \mathcal{R}(T^2)$ is an immersion taking the 1–dimensional strata of $\mathcal{R}_h(K)$ into the smooth part of $\mathcal{R}(T^2)$.

The points on the abelian arc which are the limiting points of irreducible components are called *bifurcation points.*

Proof. To put the variety $\mathcal{R}(K)$ into general position one can use essentially the same admissible perturbations as in Section 5.3.4. We only need to make sure that the image of the $\gamma_i : S^1 \times D^2 \to K$ is disjoint from ∂K. This way all perturbed flat connections restrict to flat connections over ∂K and we still have the the restriction map $r : \mathcal{R}_h(K) \to \mathcal{R}(T^2)$ from the perturbed flat moduli space $\mathcal{R}_h(K)$ to the pillowcase. The complete proof is rather technical and we refer the reader to [138] for all the details.

On the other hand, the statement about the roots of the Alexander polynomial is relatively easy to see, compare with [6]. Assume for the sake of simplicity that k is a knot in S^3 and that no perturbations are needed. Every reducible representation $\alpha : \pi_1(K) \to SU(2)$ factors through $U(1) \to SU(2)$. Therefore, Zariski tangent space $H^1(K, \operatorname{ad}\alpha)$ at α can be identified with $H^1_\alpha(K, \mathfrak{u}(1)) \oplus H^1_\alpha(K, \mathbb{C})$ according to the splitting $\mathfrak{su}(2) = \mathfrak{u}(1) \oplus \mathbb{C}$, with $U(1)$ acting on $\mathbb{C}$ with weight 2. Here, H^1_α stands for the first cohomology with coefficients in the corresponding reduction of α. An easy calculation shows that $H^1_\alpha(K, \mathfrak{u}(1)) = \mathbb{R}$ for all abelian α. At a bifurcation point the dimension of the Zariski tangent space jumps from one to at least two, which is due to the non-vanishing of $H^1_\alpha(K, \mathbb{C})$ at such a point.

Let us compute $H^1_\alpha(K, \mathbb{C})$. The knot exterior K is a $K(\pi, 1)$–space for $\pi = \pi_1 K$. The group π has Wirtinger presentation with generators $a_1, \dots, a_n$ and relations $r_1, \dots, r_n$. Let $\mathbb{Z}[\pi]$ be its group ring then the $\mathbb{Z}[\pi]$–chain complex of its universal cover in degrees 0, 1, and 2 is of the form

$$\mathbb{Z}[\pi] \longleftarrow \mathbb{Z}[\pi]^n \xleftarrow{\;A\;} \mathbb{Z}[\pi]^n \longleftarrow \cdots$$

where $A = (\partial a_i / \partial r_j)$. Every representation $\alpha : \pi_1 K \to U(1)$ factors through $H_1(K, \mathbb{Z}) = \mathbb{Z}$, which is generated by a meridian m of the knot k. Let $\alpha(m) = e^{i\phi}$ then the induced representation $H_1(K, \mathbb{Z}) \to \mathbb{C}^*$ sends m to $e^{2i\phi}$. The cohomology $H^*_\alpha(K, \mathbb{C})$ is the cohomology of the complex

$$\mathbb{C}[t, t^{-1}] \longleftarrow \mathbb{C}[t, t^{-1}]^n \xleftarrow{A(t,t^{-1})} \mathbb{C}[t, t^{-1}]^n \longleftarrow \cdots$$

where $A(t, t^{-1})$ is the classical Alexander matrix, and the formal variable t should be replaced by $e^{2i\phi}$. The Alexander polynomial is the greatest common divisor of the minors of $A(t, t^{-1})$ of order $n-1$. Since $H^0_\alpha(K, \mathbb{C}) = 0$, the non-vanishing of $H^1_\alpha(K, \mathbb{C})$ at a bifurcation point ϕ implies that $\Delta_k(e^{2i\phi}) = 0$.

Remark 5.15. Conversely, one may expect that every point ϕ on the abelian arc with $\Delta_k(e^{2i\phi}) = 0$ will be a bifurcation point. This was shown to be the case by Frohman and Klassen in [101] under the assumption that the point corresponds to a simple root of the Alexander polynomial. This result continues to hold under the much weaker hypothesis that the Tristram–Levine signature function of the knot has a jump at the point, see [136] and [141]. This implies, in particular, that under either assumption, the space $\mathcal{R}^*(K)$ is non-empty.

Example 5.16. The Alexander polynomial of a trefoil knot is $t + t^{-1} - 1$. It has two roots on the unit circle, $\exp(\pi i/3)$ and $\exp(5\pi i/3)$. These correspond to two bifurcation points $\phi = \pi/6$ and $\phi = 5\pi/6$ on the arc of abelian representations, see Figure 3.2. The Alexander polynomial of the figure-eight knot equals $3 - t - t^{-1}$, so it does not have roots on the unit circle. Respectively, there are no bifurcation points in its representation variety, see Figure 3.3.

5.4.2 The Invariants

Let m be the meridian of a knot $k \subset \Sigma$ and fix an $\alpha \in [0, \pi]$. Let $\mathcal{S}_\varphi$ consist of representations $\alpha \in \mathcal{R}(T^2)$ such that $\operatorname{tr} \alpha(m) = 2\cos\varphi$. For a generic perturbation h, the intersection $r(\mathcal{R}_h(K)) \cap \mathcal{S}_\varphi$ is transversal and hence consists of finitely many points. By counting these points with proper signs as defined in [137], one obtains an integer $\#(r(\mathcal{R}_h(K)) \cap \mathcal{S}_\varphi)$. The following theorem is due to Herald [137].

Theorem 5.17. *For every $0 \le \varphi \le \pi$ such that $\Delta_k(e^{2i\varphi}) \ne 0$, the integer $\#(r(\mathcal{R}_h(K)) \cap \mathcal{S}_\varphi)$ is independent of generic admissible perturbation h and defines an integral valued invariant $h_\varphi(\Sigma, k)$ of the knot $k \subset \Sigma$. Moreover,*

$$h_\varphi(\Sigma, k) = 4\lambda(\Sigma) + \frac{1}{2}\operatorname{sign} B_k(e^{2i\varphi}),$$

where $B_k(t) = (1-t)S + (1-t^{-1})S^t$ and S is a Seifert matrix of $k \subset \Sigma$, compare with (2.7).

Proof. One starts by checking that $h_0(\Sigma, k) = 4\lambda(\Sigma)$. As φ increases from 0 to π, the integer h_φ only changes when φ passes through a bifurcation point on the arc of reducibles. By studying the local picture at a bifurcation points, one shows that $h_\varphi(\Sigma, k)$ changes by the same rule as half the signature of the matrix $B_k(e^{2i\varphi})$.

The invariant h_φ defined in Theorem 5.17 is called the *Herald–Lin invariant*. Originally, a special case of Theorem 5.17 appeared in [199]. That paper used topological methods and only covered the case of $\varphi = \pi/2$ and $\Sigma = S^3$. The invariant introduced in [199] is sometimes referred to as the *Casson–Lin invariant*. Heusener and Kroll [141] and Austin (unpublished) have found a topological proof of Theorem 5.17 in the case of $\Sigma = S^3$ and arbitrary φ.

5.5 Equivariant Casson Invariant

Let Σ be an integral homology 3–sphere with a finite cyclic group action making it into a branched cover of another integral homology sphere with branch set a knot k. Equivariant Casson invariants of Σ are defined in [57] by counting $SU(2)$-representations of $\pi_1\Sigma$ equivariant with respect to the induced action. These invariants can be identified with certain equivariant knot signatures of k. Application of this theory to various natural cyclic group actions on the links of singularities leads to geometric proofs of the Fintushel-Stern and Neumann-Wahl formulas and their extensions, and gives a closed form formula for the Floer homology of Brieskorn homology spheres. The definition of equivariant Casson invariants in [57] is gauge theoretical. Cappell, Lee and Miller in an unpublished manuscript utilized an intersection theory to define an equivariant Casson invariant for integral homology spheres acted upon by a cyclic group of odd prime order.

5.5.1 Equivariant Gauge Theory

Let Σ be an integral homology 3-sphere and $\tau : \Sigma \to \Sigma$ an orientation preserving diffeomorphism of order n. It induces a map $\tau^* : \mathcal{R}^*(\Sigma) \to \mathcal{R}^*(\Sigma)$ on irreducible $SU(2)$–representations of $\pi_1\Sigma$ with the fixed point set $\mathcal{R}^\tau(\Sigma) = \mathrm{Fix}(\tau^*)$. For any conjugacy class $[\alpha] \in \mathcal{R}^\tau(\Sigma)$ there exists a matrix $u_\alpha \in SU(2)$ such that $\tau^*\alpha = u_\alpha \alpha u_\alpha^{-1}$. The irreducibility of α implies that u_α is defined uniquely up to sign. Since $u_{g\alpha g^{-1}} = g u_\alpha g^{-1}$, there is a well-defined correspondence $[\alpha] \mapsto |\operatorname{tr} u_\alpha|$. Observe that u_α^n belongs to the stabilizer of α, so that $u_\alpha^n = \pm 1$ and hence $|\operatorname{tr} u_\alpha| = 2\cos(\pi m/n)$, for some uniquely defined integer m such that $0 \le m \le [n/2]$. Thus, we have a splitting

$$\mathcal{R}^\tau(\Sigma) = \bigsqcup_{m=0}^{[n/2]} \mathcal{R}_m(\Sigma),$$

where $\mathcal{R}_m(\Sigma)$ consists of the conjugacy classes of irreducible representations $\alpha : \pi_1\Sigma \to SU(2)$ such that $\sigma^*\alpha = u\alpha u^{-1}$ for some $u \in SU(2)$ with $|\operatorname{tr} u| = 2\cos(\pi m/n)$, $m = 0, \dots, [n/2]$.

Let E be a trivialized $SU(2)$-bundle over Σ. Any endomorphism of E which lifts $\tau : \Sigma \to \Sigma$ induces an action on connections on E by pull-back.

As any two such lifts of τ differ by a gauge transformation, the induced action $\tau^* : \mathcal{B}^*(\Sigma) \to \mathcal{B}^*(\Sigma)$ is well-defined. Let $\mathcal{B}^\tau(\Sigma) \subset \mathcal{B}^*(\Sigma)$ be the fixed point set of τ^*. It can be broken into a disjoint union as follows.

For any connection α with $[\alpha] \in \mathcal{B}^\tau(\Sigma)$ there exists a lift ν_α of τ such that $\nu_\alpha^* \alpha = \alpha$. As the bundle E is trivialized, ν_α can be given in the base–fiber coordinates by the formula $\nu_\alpha(x,\xi) = (\tau(x), \rho_\alpha(x)\xi)$ where $\rho_\alpha : \Sigma \to SU(2)$ is a gauge transformation. Observe that ρ_α^n belongs to the stabilizer of α and therefore $\rho_\alpha^n = \pm 1$ by the irreducibility of α. The solution set of the equation $\rho^n = \pm 1$ in $SU(2)$ is a collection of conjugacy classes described by the equations $|\operatorname{tr}\rho| = 2\cos(\pi m/n)$ with $m = 0, \ldots, [n/2]$. The space $\mathcal{B}^\tau(\Sigma)$ splits into disjoint components according to which conjugacy class the image of ρ_α belongs,

$$\mathcal{B}^\tau(\Sigma) = \bigsqcup_{m=0}^{[n/2]} \mathcal{B}_m(\Sigma).$$

In particular, $\mathcal{R}_m(\Sigma)$ is contained in $\mathcal{B}_m(\Sigma)$, a refinement of the usual holonomy correspondence between flat connections over Σ and representations of the fundamental group of Σ. We will regard $\mathcal{R}_m(\Sigma)$ as the critical point set of the Chern-Simons function $\mathbf{cs}$ restricted to $\mathcal{B}_m(\Sigma)$. The Zariski tangent space of $\mathcal{R}_m(\Sigma)$ at a point α can be identified with the equivariant group cohomology

$$H^1_\tau(\pi_1\Sigma, \operatorname{ad}\alpha) = \operatorname{Fix}\{\, \tau^* : H^1(\pi_1\Sigma, \operatorname{ad}\alpha) \to H^1(\pi_1\Sigma, \operatorname{ad}\alpha)\,\}.$$

The representation space $\mathcal{R}_m(\Sigma)$ is called *non-degenerate* if the cohomology groups $H^1_\tau(\pi_1\Sigma, \operatorname{ad}\alpha)$ vanish for all $\alpha \in \mathcal{R}_m(\Sigma)$.

5.5.2 Definition of the Invariants

Let Σ be an integral homology 3–sphere with an orientation preserving diffeomorphism $\tau : \Sigma \to \Sigma$ of order n. The diffeomorphism τ induces a $\mathbb{Z}/n$–action on Σ whose quotient is always homeomorphic to a rational homology sphere, which we denote by Σ'. From this point on we will assume that Σ' is an integral homology sphere, which is equivalent to saying that τ has a fixed point on Σ. This condition ensures that Σ' has a distinguished branch set, a knot k, corresponding to the non–empty fixed point set of τ.

Refining the construction of the Floer index outlined in Section 5.3.2, we associate with any non-degenerate representation $\alpha \in \mathcal{R}^\tau(\Sigma)$ an *equivariant Floer index* as follows. Let us fix a Riemannian metric on Σ which is invariant with respect to τ, and a lift $\nu : E \to E$ of τ such that $\nu^*\alpha = \alpha$. Choose a path $\alpha(t)$, $t \in \mathbb{R}$, of connections such that $\alpha(t) = \theta$ near $-\infty$, $\alpha(t) = \alpha$ near $+\infty$, and $\nu^*\alpha(t) = \alpha(t)$ for all t (the latter can be achieved by averaging). Consider the path of self-adjoint Fredholm operators

$$K^\nu_{\alpha(t)} : (\Omega^0 \,\oplus\, \Omega^1)_\nu(\Sigma, \mathfrak{su}(2)) \to (\Omega^0 \,\oplus\, \Omega^1)_\nu(\Sigma, \mathfrak{su}(2))$$

obtained by restricting the operators (5.9) onto the differential forms invariant with respect to the induced action of ν. The Floer index $\mu^\tau(\alpha)$ is then defined to be the spectral flow of $K^\nu_{\alpha(t)}$ reduced mod 4.

Note that $\mu^\tau(\alpha) = \mu(\alpha)$ mod 2 whenever n is odd, see [59]. In general, this is not true if n is even, see Example 5.24.

If $\mathcal{R}^\tau(\Sigma)$ is non-degenerate, the *equivariant Casson invariant* is defined by the formula

$$\lambda^\tau(\Sigma) = 1/2 \sum_{\alpha\in\mathcal{R}^\tau(\Sigma)} (-1)^{\mu^\tau(\alpha)}. \tag{5.16}$$

In addition, *refined equivariant Casson invariants* are defined as

$$\lambda^\tau_m(\Sigma) = \kappa/2 \sum_{\alpha\in\mathcal{R}_m(\Sigma)} (-1)^{\mu^\tau(\alpha)}, \tag{5.17}$$

where $\kappa = 1/2$ if $m/n \neq 1/2$ and $\kappa = 1$ if $m/n = 1/2$. The extra factor of κ is explained by the fact that the correspondence $\alpha \mapsto \operatorname{tr} u_\alpha$ is only well-defined up to a sign. Because of that, the count of representations with fixed $|\operatorname{tr} u_\alpha|$ is roughly twice the count of those with fixed $\operatorname{tr} u_\alpha$, unless $\operatorname{tr} u_\alpha = 0$.

If $\mathcal{R}^\tau(\Sigma)$ fails to be non-degenerate, an approach similar to that of Section 5.3.4 and Section 5.4.1 provides a family of generic admissible perturbation functions $h : \mathcal{B}^\tau(\Sigma) \to \mathbb{R}$ which yield a non-degenerate equivariant representation space, see [57] for details. The equivariant Casson invariant is then defined by counting points in this space.

5.5.3 Equivariant Casson and Knot Signatures

The equivariant Casson invariants of a homology sphere with a cyclic group action are expressed in [57] in terms of equivariant knot signatures of the branch set of the action. The equivariant knot signatures $\operatorname{sign}^{m/n}(k)$ are defined as in (2.7).

Theorem 5.18. *Let Σ be an integral homology 3–sphere and $\tau : \Sigma \to \Sigma$ an orientation preserving diffeomorphism of order n. Suppose that the quotient of Σ by the induced $\mathbb{Z}/n$–action is homeomorphic to an integral homology sphere Σ' with branch set a knot $k \subset \Sigma'$. Then, for any $m = 0, \dots, [n/2]$,*

$$\lambda^\tau_m(\Sigma) = \lambda(\Sigma') + \frac{1}{8} \cdot \operatorname{sign}^{m/n}(k),$$

and therefore, for the (global) equivariant Casson invariant,

$$\lambda^\tau(\Sigma) = n \cdot \lambda(\Sigma') + \frac{1}{8} \cdot \sum_{k=0}^{n-1} \operatorname{sign}^{m/n}(k).$$

Proof. A non-degenerate equivariant representation $\alpha : \pi_1 \Sigma \to SU(2)$ such that $\tau^* \alpha = u \alpha u^{-1}$ can be pushed down to the quotient $\Sigma/\tau = \Sigma'$, away from the branch set k, to give a representation $\alpha' : \pi_1(\Sigma' \setminus k) \to SU(2)$ with $\alpha'(m)$ conjugate to u (where m is the meridian of k). According to Theorem 5.17, a certain count of representations α' with a fixed trace of the meridians essentially gives the sum of an equivariant signature of k and the Casson invariant of Σ'. Therefore, to prove the theorem, one compares the count of representations α given by (5.16) with that of Theorem 5.17. This is done with the help of the orbifold gauge theory developed in [59], see also Section 6.7.5. In the degenerate case, special attention should be paid to matching perturbations in the two theories. For a complete proof, see [57].

5.5.4 Applications

Let k be a knot in an integral homology 3–sphere Σ, which is the boundary of a smooth oriented 4–manifold M with $H_1(M, \mathbb{Z}) = 0$. Let Σ_n be the n–fold cyclic branched cover of Σ with branch set the knot k. Given a Seifert surface F of the knot k, let M_n be the n–fold cyclic branched cover of M whose branch set is the surface F with its interior pushed slightly into M. Then M_n is a smooth 4–dimensional manifold with boundary Σ_n and with a natural $\mathbb{Z}/n$–action. Fix a generator $\tau : M_n \to M_n$ of $\mathbb{Z}/n$.

Although the manifold M_n might depend on the choice of F, this will not be the case for its signature. More precisely, the vector space $H_2(M_n, \mathbb{C})$ can be split as $H_2(M_n, \mathbb{C}) = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \dots \mathcal{H}_{n-1}$, where $\mathcal{H}_m$ is the eigenspace of the operator $\tau_* : H_2(M_n, \mathbb{C}) \to H_2(M_n, \mathbb{C})$ with eigenvalue $e^{2\pi i m/n}$. The signature of the intersection form of M_n restricted to $\mathcal{H}_m$ is denoted by $\operatorname{sign}^m(M_n)$ and is called the mth *equivariant signature* of M_n. Obviously,

$$\operatorname{sign}(M_n) = \sum_{m=0}^{n-1} \operatorname{sign}^m(M_n),$$

and it is a classical result of Viro (see Theorem 4.4 and Section 4.8 in [301]) that $\operatorname{sign}^m(M_n) = \operatorname{sign}(M) + \operatorname{sign}^{m/n}(k)$. This fact together with Theorem 5.18 proves the following result.

Corollary 5.19. *Suppose that the 3–manifold $\Sigma_n = \partial M_n$ constructed above from a knot $k \subset \Sigma = \partial M$ is an integral homology sphere. Then*

$$\lambda^\tau(\Sigma_n) - n \cdot \lambda(\Sigma) = \frac{1}{8}\left(\operatorname{sign}(M_n) - n \cdot \operatorname{sign}(M)\right).$$

Similarly, for the refined equivariant Casson invariants (5.17)

$$\lambda^\tau_m(\Sigma_n) - \lambda(\Sigma) = \frac{1}{8}\left(\operatorname{sign}^m(M_n) - \operatorname{sign}(M)\right).$$

Example 5.20. Let $p, q, r \geq 2$ be pairwise relatively prime integers then the link $\Sigma(p, q, r)$ of singularity (1.1) is an integral homology sphere. The map $\tau(x, y, z) = (x, y, e^{2\pi i/r} z)$ is an orientation-preserving diffeomorphism of order r, which makes $\Sigma(p, q, r)$ into an r–fold cyclic branched cover of S^3 with branch set a (p, q)–torus knot. According to Corollary 5.19,

$$\lambda^\tau(\Sigma(p, q, r)) = \frac{1}{8} \operatorname{sign} M(p, q, r),$$

where $M(p, q, r)$ is the Milnor fiber (for a proper choice of the branching surface). Theorem 5.23 below implies that in fact $\lambda^\tau(\Sigma(p, q, r)) = \lambda(\Sigma(p, q, r))$, so that we arrive at the Fintushel–Stern formula, see Theorem 3.29. The second part of Corollary 5.19 refines this formula in that it relates the equivariant signatures of $M(p, q, r)$ to the invariants $\lambda^\tau_m(\Sigma(p, q, r))$.

Example 5.21. Let $a_1, \ldots, a_n$ be pairwise relatively prime integers, and let $\Sigma(a_1, \ldots, a_n)$ be the link of singularity (1.2). Then $\Sigma(a_1, \ldots, a_n)$ is an integral homology sphere which can be represented as an a_n–fold branched cover of integral homology sphere $\Sigma(a_1, \ldots, a_{n-1})$ via the action $\tau(z_1, \ldots, z_{n-1}, z_n) = (z_1, \ldots, z_{n-1}, e^{2\pi i/a_n} z_n)$. The branch set of this covering is a regular fiber of the Seifert fibration of $\Sigma(a_1, \ldots, a_{n-1})$. According to Theorem 5.23 below, invariants λ^τ and λ coincide in this case, and an induction by n then proves that

$$\lambda(\Sigma(a_1, \ldots, a_n)) = \frac{1}{8} \operatorname{sign} M(a_1, \ldots, a_n),$$

where $M(a_1, \ldots, a_n)$ is the Milnor fiber, compare with Theorem 3.32.

Example 5.22. Let Σ be an integral homology sphere obtained as the link of singularity at zero of $f(x, y) + z^n = 0$, see Example 1.19. The map $\tau(x, y, z) = (x, y, e^{2\pi i/n} z)$ is an orientation preserving diffeomorphism of Σ of order n, which makes Σ into a cyclic branched cover of S^3 with branch set a graph knot, compare with Example 1.23. The invariants $\lambda^\tau(\Sigma)$ and $\lambda(\Sigma)$ again coincide, see Theorem 5.23, therefore, $\lambda(\Sigma)$ equals one eighth of the Milnor fiber signature.

The last three examples provide geometric proofs of the Neumann-Wahl conjecture, see Section 3.5.3, for several families of algebraic links. The conjecture was originally verified for all these families in [83] and [243] by purely combinatorial methods. The crucial fact used in the above examples is of course the equality $\lambda = \lambda^\tau$. This equality is a consequence of the following theorem proved in [57] (for the definition of graph knot see Section 1.1.8).

Theorem 5.23. *Let $k \subset \Sigma$ be a graph knot in an integral homology sphere and let Σ_n be the n–fold cyclic branched cover of Σ with branch set the knot k. If Σ_n is an integral homology sphere then $\lambda^\tau(\Sigma_n) = \lambda(\Sigma_n)$.*

The following examples demonstrate that Theorem 5.23 is specific to graph knots and that in general the equivariant Casson invariant need not be equal to the Casson invariant.

Example 5.24. Let $\Sigma(p,q,r)$ be a Brieskorn homology sphere (1.1) and consider the complex conjugation involution $\tau(x,y,z)=(\bar{x},\bar{y},\bar{z})$. The quotient of $\Sigma(p,q,r)$ by this $\mathbb{Z}/2$–action is S^3, with branch set a Montesinos knot of type (p,q,r), see Example 1.22. The induced action of τ on the representation space $\mathcal{R}^*(\Sigma(p,q,r))$ is identity. More precisely, for every representation $\alpha:\pi_1(\Sigma(p,q,r))\to SU(2)$, there exists $u\in SU(2)$ such that $\operatorname{tr} u=0$ and $\tau^*\alpha=u\alpha u^{-1}$. According to [272], the Floer indices are related by the formula $\mu(\alpha)=2\mu^\tau(\alpha)+1 \bmod 4$. Therefore, the equivariant Casson invariant $\lambda^\tau(\Sigma(p,q,r))$ is obtained by counting all the representations in $\mathcal{R}^*(\Sigma(p,q,r))$ but with signs different from those giving $\lambda(\Sigma(p,q,r))$. This observation was used in [272] to give a closed form formula for the Floer homology of $\Sigma(p,q,r)$, see also Section 6.4.2. These results were extended to all Seifert fibered homology spheres $\Sigma(a_1,\dots,a_n)$ in [278].

Example 5.25. Every plumbed integral homology sphere Σ admits a preferred involution $\tau:\Sigma\to\Sigma$ generalizing the complex conjugation involution on links of singularities, see Example 1.26. Let $\lambda^\tau(\Sigma)$ be the corresponding equivariant Casson invariant then $\lambda^\tau(\Sigma)=\bar{\mu}(\Sigma)$, the $\bar{\mu}$–invariant of Neumann and Siebenmann, see Section 7.2.3.

Example 5.26. Let k be a knot in S^3 and denote by $D_\varepsilon k$ its untwisted Whitehead double with ε–clasp, see Figure 3.6. Let Σ be the n-fold cyclic branched cover of S^3 with branch set $D_\varepsilon k$. Then Σ is an integral homology sphere and $\lambda^\tau(\Sigma)=0$ for any knot k because all the equivariant signatures of $D_\varepsilon k$ are equal to zero. On the other hand, $\lambda(\Sigma)=n\varepsilon\cdot\Delta''_k(1)$ according to Theorem 3.23. Therefore, in general, $\lambda(\Sigma)\neq\lambda^\tau(\Sigma)$.

Example 5.27. Let Σ be an integral homology sphere represented as an n–fold cyclic branched cover of S^3 with branch set a knot k. If $n=2$, the difference between the Casson and equivariant Casson invariants can be measured in terms of the Jones polynomial of k, see Theorem 3.21 (where $(1/8)\cdot\operatorname{sign} k=\lambda^\tau(\Sigma_2)$). For arbitrary n, this difference can be expressed in terms of residues of certain rational functions related to the Kontsevich integral of k, see [122].

5.6 The $SU(3)$ Casson Invariant

This section describes the $SU(3)$ Casson invariant defined by H. Boden and C. Herald in [30]. Their definition extends the gauge theoretical approach to the Casson invariant outlined in Section 5.3. Unlike the regular Casson invariant, the invariant of Boden and Herald takes real values. Recently, two integer valued versions of the $SU(3)$ Casson invariant were announced, one by Cappell, Lee and Miller [46] and the other by Boden, Herald and Kirk [31]. An approach to defining $SU(n)$ Casson invariants for all n was proposed in [49].

5.6.1 Some $SU(3)$–Gauge Theory

Let Σ be an oriented integral homology 3–sphere and let $\mathcal{A}$ be the space of $SU(3)$–connections on a trivialized bundle $\Sigma \times SU(3)$. By fixing a trivial connection θ, we identify $\mathcal{A}$ with the linear space $\Omega^1(\Sigma, \mathfrak{su}(3))$ of $\mathfrak{su}(3)$–valued differential 1–forms on Σ. The gauge group $\mathcal{G} = \operatorname{Map}(\Sigma, SU(3))$ consisting of the automorphisms of the bundle $\Sigma \times SU(3)$ acts on $\mathcal{A}$ by the rule $g^*A = g^{-1}dg + g^{-1}Ag$, where $g \in \mathcal{G}$ and $A \in \mathcal{A}$, compare with Section 5.1. This action is not free, and according to the size of the stabilizer, there is a natural stratification of $\mathcal{A}$ and also of the quotient space $\mathcal{B} = \mathcal{A}/\mathcal{G}$. An $SU(3)$–connection A is called *irreducible* if its stabilizer consists of constant maps into $\mathbb{Z}/3$, the center of $SU(3)$. Irreducible connections form the top stratum $\mathcal{A}^*$ and its quotient $\mathcal{B}^* = \mathcal{A}^*/\mathcal{G}$ has the structure of a Banach manifold (in proper Sobolev completions). Below the top stratum, there are strata whose stabilizers are respectively $U(1)$, $S(U(1) \times U(1) \times U(1))$, $S(U(1) \times U(2))$, and $SU(3)$.

Let us consider the space $\mathcal{R}(\Sigma)$ of the gauge equivalence classes of flat $SU(3)$ connections. As usual, it can be identified with the $SU(3)$ character variety

$$\mathcal{R}(\Sigma) = \operatorname{Hom}(\pi_1(\Sigma), SU(3))/\, SU(3)$$

consisting of $SU(3)$ representations of $\pi_1(\Sigma)$ modulo conjugation. The only strata relevant to $\mathcal{R}(\Sigma)$ are those with stabilizers $\mathbb{Z}/3, U(1)$, and $SU(3)$. The reason is that, for the integral homology sphere Σ, there exist no non-trivial representations $\pi_1(\Sigma) \to U(1)$. According to the three types of stabilizers, $\mathbb{Z}/3$, $U(1)$, and $SU(3)$, the space $\mathcal{R}(\Sigma)$ splits into a disjoint union $\mathcal{R}(\Sigma) = \mathcal{R}^*(\Sigma) \cup \mathcal{R}^a(\Sigma) \cup \{\theta\}$.

5.6.2 Definition of the Invariant

As in the $SU(2)$ case, the space $\mathcal{R}(\Sigma)$ can be viewed as the critical point set of the Chern–Simons function $\mathbf{cs} : \mathcal{B} \to \mathbb{R}/\mathbb{Z}$, see Section 5.3.2. The Chern–Simons function may have to be perturbed, which is done with the help of admissible perturbation functions $h : \mathcal{B} \to \mathbb{R}$. These functions are modeled after admissible perturbation functions (5.12); their precise definition can be found in [29] and [30]. Let

$$\mathcal{R}_h(\Sigma) = \{\, A \mid\, * F_A - 4\pi^2\, (\nabla h)(A) = 0 \,\}/\mathcal{G}$$

be the perturbed flat moduli space. Boden and Herald [30] show that there exists an $\varepsilon > 0$ such that, for all admissible perturbation functions h with $\|h\| \le \varepsilon$, the space $\mathcal{R}_h(\Sigma)$ splits as

$$\mathcal{R}_h(\Sigma) = \mathcal{R}^*_h(\Sigma) \cup \mathcal{R}^a_h(\Sigma) \cup \{\theta\},$$

where $\mathcal{R}^*_h(\Sigma)$ and $\mathcal{R}^a_h(\Sigma)$ are compact 0–dimensional manifolds consisting of gauge orbits that satisfy certain cohomological non-degeneracy condition, see [29] or [30]. Moreover, for any perturbed flat connection A there exists a unique component $\hat{A}$ of flat connections which is within ε–distance of A.

Given an $SU(3)$–connection A, define a self–adjoint elliptic operator K_A as in (5.9) with only difference that now it will act on the space $\Omega^0(\Sigma, \mathfrak{su}(3)) \oplus \Omega^1(\Sigma, \mathfrak{su}(3))$. Let A be a flat connection, and choose a path $A_t \in \mathcal{A}$ with $A_0 = \theta$ and $A_1 = A$. Define $\mathrm{sf}\,(\theta, A)$ to be the spectral flow of the path K_{A_t}, see Section 5.3.2. If $[A] \in \mathcal{R}^a(\Sigma)$ then we can choose the path so that each A_t has stabilizer $U(1)$ for all $0 < t \leq 1$. Adjusting by a path of gauge transformations, we can assume that, for all $0 \leq t \leq 1$,

$$A_t \in \mathcal{A}_{S(U(2)\times U(1))},$$

the space of connections on $\Sigma \times S(U(2) \times U(1))$. Let $\mathfrak{h} = \mathfrak{s}(\mathfrak{u}(2) \times \mathfrak{u}(1))$ be the Lie algebra of $S(U(2) \times U(1))$ viewed as a Lie subalgebra of $\mathfrak{su}(3)$. Then $\mathfrak{su}(3) = \mathfrak{h} \oplus \mathbb{C}^2$, and the spectral flow decomposes as

$$\mathrm{sf}\,(\theta, A) = \mathrm{sf}_{\,\mathfrak{h}}(\theta, A) + \mathrm{sf}_{\,\mathbb{C}^2}(\theta, A).$$

All the above can be perturbed if necessary by using an admissible perturbation function h. The operator K_A will then be replaced by the operator $K_{A,h}$, see (5.13), and the above spectral flows will be the spectral flows of families of the $K_{A,h}$. For a sufficiently small admissible perturbation h define

$$\lambda'_{BH}(\Sigma) = \sum_{[A]\in\mathcal{R}^*_h(\Sigma)} (-1)^{\mathrm{sf}\,(\theta,A)}$$

and

$$\lambda''_{BH}(\Sigma) = \sum_{[A]\in\mathcal{R}^a_h(\Sigma)} (-1)^{\mathrm{sf}\,(\theta,A)} \cdot \left(\mathrm{sf}_{\,\mathbb{C}^2}(\theta, A) - 4\,\mathbf{cs}(\hat{A}) + 2\right),$$

where $\hat{A}$ is the unique component of flat connections within $\|h\|$–distance of A, see above. The following theorem is proved in [29].

Theorem 5.28. *For a generic sufficiently small admissible perturbation h, the quantity $\lambda_{BH}(\Sigma) = \lambda'_{BH}(\Sigma) - 1/2 \cdot \lambda''_{BH}(\Sigma)$ is an invariant of integral homology 3–sphere Σ.*

The invariant $\lambda_{BH}(\Sigma)$ is called the $SU(3)$ *Casson invariant* of Σ. The number $\lambda'_{BH}(\Sigma)$ may look like the natural generalization of the Casson invariant, compare with (5.11). Unfortunately, it depends on the choice of perturbation h, which is due to the fact that reducible connections can perturb into irreducible ones. A similar situation occurs in Walker's generalization of the Casson invariant, see Example 4.4.

Whenever an irreducible flat connection sinks into or emerges from reducible connections, an integer jump occurs in the spectral flow $\mathrm{sf}_{\,\mathbb{C}^2}(\theta, A)$. Thus the discrepancy in $\lambda'_{BH}(\Sigma)$ is compensated by the second term,

$\lambda''_{BH}(\Sigma)$. It should be noted that only the difference $\text{sf}_{\mathbb{C}^2}(\theta, A) - 4\,\mathbf{cs}(\hat{A})$ is well defined on the gauge orbit; each term individually depends on the choice of representative for $[A]$. Adding in the constant part of the second term only adds a multiple of $SU(2)$ Casson invariant. It is needed to assure certain properties of the resulting invariant.

5.6.3 Properties and Computations

The $SU(3)$ Casson invariant λ_{BH} defined in Theorem 5.28 takes real values. The conjectured rationality of the Chern–Simons function **cs** on flat connections, see Section 5.2, would imply that λ_{BH} is rational. The invariant λ_{BH} is insensitive to the change of orientation,

$$\lambda_{BH}(-\Sigma) = \lambda_{BH}(\Sigma).$$

It satisfies the following connected sum formula, see [29],

$$\lambda_{BH}(\Sigma_1 \# \Sigma_2) = \lambda_{BH}(\Sigma_1) + \lambda_{BH}(\Sigma) + 4\,\lambda(\Sigma_1)\,\lambda(\Sigma_2),$$

where λ is the original Casson invariant. Although λ_{BH} is not additive under the connected sum operation, the difference $\lambda_{BH} - 2\,\lambda^2$ is. If $\lambda_{BH}(\Sigma) \neq 0$ then $\pi_1(\Sigma)$ admits a non-trivial representation into $SU(3)$ or $SU(2)$.

The invariant λ_{BH} is rather difficult to compute because one lacks algebraic tools such as the surgery formula, which are so useful in computing the $SU(2)$ Casson invariant. The following calculations can be found in [32].

Example 5.29. Let $\Sigma_k(p,q)$ be the integral homology sphere obtained by $(1/k)$–surgery on the right handed (p,q)–torus knot so that $\Sigma_{-k}(p,q) = \Sigma(p,q,pqk+1)$ and $\Sigma_k(p,q) = -\Sigma(p,q,pqk-1)$ for $k \geq 0$. Then

$$\begin{aligned}
\lambda_{BH}(\Sigma_k(2,3)) &= \frac{84k^3 - 138k^2 + 19k}{6(6k-1)},\\
\lambda_{BH}(\Sigma_k(2,5)) &= \frac{3100k^3 - 2850k^2 + 241k}{10(10k-1)},\\
\lambda_{BH}(\Sigma_k(2,7)) &= \frac{26264k^3 - 16156k^2 + 970k}{14(14k-1)},\\
\lambda_{BH}(\Sigma_k(2,9)) &= \frac{124200k^3 - 59004k^2 + 2758k}{18(18k-1)}.
\end{aligned}$$

Note that the $SU(3)$ Casson invariants of the manifolds obtained by $(1/k)$–surgery and $(-1/k)$–surgery on the same torus knot have different absolute values while their $SU(2)$ Casson invariants only differ in sign.