Chapter 1

Introduction

1.1. Introduction

Combinatorial optimization searches for an optimum object in a finite collection of objects. Typically, the collection has a concise representation (like a graph), while the number of objects is huge — more precisely, grows exponentially in the size of the representation (like all matchings or all Hamiltonian circuits). So scanning all objects one by one and selecting the best one is not an option. More efficient methods should be found.

In the 1960s, Edmonds advocated the idea to call a method efficient if its running time is bounded by a polynomial in the size of the representation. Since then, this criterion has won broad acceptance, also because Edmonds found polynomial-time algorithms for several important combinatorial optimization problems (like the matching problem). The class of polynomial-time solvable problems is denoted by P.

Further relief in the landscape of combinatorial optimization was discovered around 1970 when Cook and Karp found out that several other prominent combinatorial optimization problems (including the traveling salesman problem) are the hardest in a large natural class of problems, the class NP. The class NP includes most combinatorial optimization problems. Any problem in NP can be reduced to such 'NP-complete' problems. All NP-complete problems are equivalent in the sense that the polynomial-time solvability of one of them implies the same for all of them.

Almost every combinatorial optimization problem has since been either proved to be polynomial-time solvable or NP-complete — and none of the problems have been proved to be both. This spotlights the big mystery: are the two properties disjoint (equivalently, P≠NP), or do they coincide (P=NP)?

This book focuses on those combinatorial optimization problems that have been proved to be solvable in polynomial time, that is, those that have been proved to belong to P. Next to polynomial-time solvability, we focus on the related polyhedra and min-max relations.

These three aspects have turned out to be closely related, as was shown also by Edmonds. Often a polynomial-time algorithm yields, as a by-product, a description (in terms of inequalities) of an associated polyhedron. Conversely, an appropriate description of the polyhedron often implies the polynomial-time solvability of the associated optimization problem, by applying linear programming techniques. With the duality theorem of linear programming, polyhedral characterizations yield min-max relations, and vice versa.

So the span of this book can be portrayed alternatively by those combinatorial optimization problems that yield well-described polyhedra and min-max relations. This field of discrete mathematics is called *polyhedral combinatorics*. In the following sections we give some basic, illustrative examples.[1]

1.2. Matchings

Let $G = (V, E)$ be an undirected graph and let $w : E \to \mathbb{R}_+$. For any subset F of E, denote

$$(1.1) \qquad w(F) := \sum_{e \in F} w(e).$$

We will call $w(F)$ the *weight* of F.

Suppose that we want to find a *matching* (= set of disjoint edges) M in G with weight $w(M)$ as large as possible. In notation, we want to 'solve'

$$(1.2) \qquad \max\{w(M) \mid M \text{ matching in } G\}.$$

We can formulate this problem equivalently as follows. For any matching M, denote the incidence vector of M in $\mathbb{R}^E$ by χ^M; that is,

$$(1.3) \qquad \chi^M(e) := \begin{cases} 1 & \text{if } e \in M, \\ 0 & \text{if } e \notin M, \end{cases}$$

for $e \in E$. Considering w as a *vector* in $\mathbb{R}^E$, we have $w(M) = w^{\mathsf{T}}\chi^M$. Hence problem (1.2) can be rewritten as

$$(1.4) \qquad \max\{w^{\mathsf{T}}\chi^M \mid M \text{ matching in } G\}.$$

This amounts to maximizing the linear function $w^{\mathsf{T}}x$ over a finite set of vectors. Therefore, the optimum value does not change if we maximize over the *convex hull* of these vectors:

$$(1.5) \qquad \max\{w^{\mathsf{T}}x \mid x \in \text{conv.hull}\{\chi^M \mid M \text{ matching in } G\}\}.$$

The set

$$(1.6) \qquad \text{conv.hull}\{\chi^M \mid M \text{ matching in } G\}$$

is a polytope in $\mathbb{R}^E$, called the *matching polytope* of G. As it is a polytope, there exist a matrix A and a vector b such that

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[1] Terms used but not introduced yet can be found later in this book — consult the Subject Index.

$$\text{conv.hull}\{\chi^M \mid M \text{ matching in } G\} = \{x \in \mathbb{R}^E \mid x \geq \mathbf{0}, Ax \leq b\}. \tag{1.7}$$

Then problem (1.5) is equivalent to

$$\max\{w^{\mathsf{T}}x \mid x \geq \mathbf{0}, Ax \leq b\}. \tag{1.8}$$

In this way we have formulated the original combinatorial problem (1.2) as a *linear programming* problem. This enables us to apply linear programming methods to study the original problem.

The question at this point is, however, how to find the matrix A and the vector b. We know that A and b do exist, but we must know them in order to apply linear programming methods.

If G is bipartite, it turns out that the matching polytope of G is equal to the set of all vectors $x \in \mathbb{R}^E$ satisfying

$$\begin{aligned} x(e) &\geq 0 && \text{for } e \in E, \\ \sum_{e \ni v} x(e) &\leq 1 && \text{for } v \in V. \end{aligned} \tag{1.9}$$

(The sum ranges over all edges e containing v.) That is, for A we can take the $V \times E$ incidence matrix of G and for b the all-one vector $\mathbf{1}$ in $\mathbb{R}^V$.

It is not difficult to show that the matching polytope for bipartite graphs is indeed completely determined by (1.9). First note that the matching polytope is contained in the polytope determined by (1.9), since χ^M satisfies (1.9) for each matching M. To see the reverse inclusion, we note that, if G is bipartite, then the matrix A is *totally unimodular*, i.e., each square submatrix has determinant belonging to $\{0, +1, -1\}$. (This easy fact will be proved in Section 18.2.) The total unimodularity of A implies that the vertices of the polytope determined by (1.9) are *integer* vectors, i.e., belong to $\mathbb{Z}^E$. Now each integer vector satisfying (1.9) must trivially be equal to χ^M for some matching M. Hence, if G is bipartite, the matching polytope is determined by (1.9).

We therefore can apply linear programming techniques to handle problem (1.2). Thus we can find a maximum-weight matching in a bipartite graph in polynomial time, with any polynomial-time linear programming algorithm. Moreover, the duality theorem of linear programming gives

$$\begin{aligned} &\max\{w(M) \mid M \text{ matching in } G\} = \max\{w^{\mathsf{T}}x \mid x \geq \mathbf{0}, Ax \leq \mathbf{1}\} \\ &= \min\{y^{\mathsf{T}}\mathbf{1} \mid y \geq \mathbf{0}, y^{\mathsf{T}}A \geq w^{\mathsf{T}}\}. \end{aligned} \tag{1.10}$$

If we take for w the all-one vector $\mathbf{1}$ in $\mathbb{R}^E$, we can derive from this Kőnig's matching theorem (Kőnig [1931]):

(1.11) the maximum size of a matching in a bipartite graph is equal to the minimum size of a vertex cover,

where a *vertex cover* is a set of vertices intersecting each edge. Indeed, the left-most expression in (1.10) is equal to the maximum size of a matching. The minimum can be seen to be attained by an integer vector y, again by the total unimodularity of A. This vector y is a $0,1$ vector in $\mathbb{R}^V$, and hence is the incidence vector χ^U of some subset U of V. Then $y^{\mathsf{T}}A \geq \mathbf{1}^{\mathsf{T}}$ implies that U is a vertex cover. Therefore, the right-most expression is equal to the minimum size of a vertex cover.

Kőnig's matching theorem (1.11) is an example of a *min-max formula* that can be derived from a polyhedral characterization. Conversely, min-max formulas (in particular in a weighted form) often give polyhedral characterizations.

The polyhedral description together with linear programming duality also gives a *certificate* of optimality of a matching M: to convince your 'boss' that a certain matching M has maximum size, it is possible and sufficient to display a vertex cover of size $|M|$. In other words, it yields a *good characterization* for the maximum-size matching problem in bipartite graphs.

1.3. But what about nonbipartite graphs?

If G is *nonbipartite*, the matching polytope is not determined by (1.9): if C is an odd circuit in G, then the vector $x \in \mathbb{R}^E$ defined by $x(e) := \frac{1}{2}$ if $e \in EC$ and $x(e) := 0$ if $e \notin EC$, satisfies (1.9) but does not belong to the matching polytope of G.

A pioneering and central theorem in polyhedral combinatorics of Edmonds [1965b] gives a complete description of the inequalities needed to describe the matching polytope for arbitrary graphs: one should add to (1.9) the inequalities

$$(1.12) \qquad \sum_{e \subseteq U} x(e) \leq \lfloor \tfrac{1}{2}|U| \rfloor \text{ for each odd-size subset } U \text{ of } V.$$

Trivially, the incidence vector χ^M of any matching M satisfies (1.12). So the matching polytope of G is contained in the polytope determined by (1.9) and (1.12). The content of Edmonds' theorem is the converse inclusion. This will be proved in Chapter 25.

In fact, Edmonds designed a polynomial-time algorithm to find a maximum-weight matching in a graph, which gave this polyhedral characterization as a by-product. Conversely, from the characterization one may derive the polynomial-time solvability of the weighted matching problem, with the ellipsoid method. In applying linear programming methods for this, one will be faced with the fact that the system $Ax \leq b$ consists of exponentially many inequalities, since there exist exponentially many odd-size subsets U of V. So in order to solve the problem with linear programming methods, we cannot just list all inequalities. However, the ellipsoid method does not require that all inequalities are listed a priori. It suffices to have a polynomial-time algorithm answering the question:

(1.13) given $x \in \mathbb{R}^E$, does x belong to the matching polytope of G?

Such an algorithm indeed exists, as it has been shown that the inequalities (1.9) and (1.12) can be checked in time bounded by a polynomial in $|V|$, $|E|$, and the size of x. This method obviously should avoid testing all inequalities (1.12) one by one.

Combining the description of the matching polytope with the duality theorem of linear programming gives a min-max formula for the maximum weight of a matching. It again yields a certificate of optimality: if we have a matching M, we can convince our 'boss' that M has maximum weight, by supplying a dual solution y of objective value $w(M)$. So the maximum-weight matching problem has a good characterization — i.e., belongs to NP∩co-NP.

This gives one motivation for studying polyhedral methods. The ellipsoid method proves polynomial-time solvability, it however does not yield a practical method, but rather an incentive to search for a practically efficient algorithm. The polyhedral method can be helpful also in this, e.g., by imitating the simplex method with a constraint generation technique, or by a primal-dual approach.

1.4. Hamiltonian circuits and the traveling salesman problem

As we discussed above, matching is an area where the search for an inequality system determining the corresponding polytope has been successful. This is in contrast with, for instance, Hamiltonian circuits. No full description in terms of inequalities of the convex hull of the incidence vectors of Hamiltonian circuits — the *traveling salesman polytope* — is known. The corresponding optimization problem is the traveling salesman problem: 'find a Hamiltonian circuit of minimum weight', which problem is NP-complete. This implies that, unless NP=co-NP, there exist facet-inducing inequalities for the traveling salesman polytope that have no polynomial-time certificate of validity. Otherwise, linear programming duality would yield a good characterization. So unless NP=co-NP there is no hope for an appropriate characterization of the traveling salesman polytope.

Moreover, unless NP=P, there is no polynomial-time algorithm answering the question

(1.14) given $x \in \mathbb{R}^E$, does x belong to the traveling salesman polytope?

Otherwise, the ellipsoid method would give the polynomial-time solvability of the traveling salesman problem.

Nevertheless, polyhedral combinatorics can be applied to the traveling salesman problem in a positive way. If we include the traveling salesman polytope in a larger polytope (a *relaxation*) over which we *can* optimize in polynomial time, we obtain a polynomial-time computable bound for the traveling salesman problem. The closer the relaxation is to the traveling salesman polytope, the better the bound is. This can be very useful in a branch-and-bound algorithm. This idea originates from Dantzig, Fulkerson, and Johnson [1954b].

1.5. Historical and further notes

1.5a. Historical sketch on polyhedral combinatorics

The first min-max relations in combinatorial optimization were proved by Dénes Kőnig [1916,1931], on edge-colouring and matchings in bipartite graphs, and by Karl Menger [1927], on disjoint paths in graphs. The matching theorem of Kőnig was extended to the weighted case by Egerváry [1931]. The proofs by Kőnig and Egerváry were in principal algorithmic, and also for Menger's theorem an algorithmic proof was given in the 1930s. The theorem of Egerváry may be seen as polyhedral.

Applying linear programming techniques to combinatorial optimization problems came along with the introduction of linear programming in the 1940s and 1950s. In fact, linear programming forms the hinge in the history of combinatorial optimization. Its initial conception by Kantorovich and Koopmans was motivated by combinatorial applications, in particular in transportation and transshipment.

After the formulation of linear programming as generic problem, and the development in 1947 by Dantzig of the simplex method as a tool, one has tried to attack about all combinatorial optimization problems with linear programming techniques, quite often very successfully. In the 1950s, Dantzig, Ford, Fulkerson, Hoffman, Kuhn, and others studied problems like the transportation, maximum flow, and assignment problems. These problems can be reduced to linear programming by the total unimodularity of the underlying matrix, thus yielding extensions and polyhedral and algorithmic interpretations of the earlier results of Kőnig, Egerváry, and Menger. Kuhn realized that the polyhedral methods of Egerváry for weighted bipartite matching are in fact algorithmic, and yield the efficient 'Hungarian' method for the assignment problem. Dantzig, Fulkerson, and Johnson gave a solution method for the traveling salesman problem, based on linear programming with a rudimentary, combinatorial version of a cutting plane technique.

A considerable extension and deepening, and a major justification, of the field of polyhedral combinatorics was obtained in the 1960s and 1970s by the work and pioneering vision of Jack Edmonds. He characterized basic polytopes like the matching polytope, the arborescence polytope, and the matroid intersection polytope; he introduced (with Giles) the important concept of total dual integrality; and he advocated the interconnections between polyhedra, min-max relations, good characterizations, and efficient algorithms. We give a few quotes in which Edmonds enters into these issues.

In his paper presenting a maximum-size matching algorithm, Edmonds [1965d] gave a polyhedral argument why an algorithm can lead to a min-max theorem:

> It is reasonable to hope for a theorem of this kind because any problem which involves maximizing a linear form by one of a discrete set of non-negative vectors has associated with it a dual problem in the following sense. The discrete set of vectors has a convex hull which is the intersection of a discrete set of half-spaces. The value of the linear form is as large for some vector of the discrete set

> as it is for any other vector in the convex hull. Therefore, the discrete problem is equivalent to an ordinary linear programme whose constraints, together with non-negativity, are given by the half-spaces. The dual (more precisely, a dual) of the discrete problem is the dual of this ordinary linear programme.
>
> For a class of discrete problems, formulated in a natural way, one may hope then that equivalent linear constraints are pleasant enough though they are not explicit in the discrete formulation.

In another paper (characterizing the matching polytope), Edmonds [1965b] stressed that the number of inequalities is not relevant:

> The results of this paper suggest that, in applying linear programming to a combinatorial problem, the number of relevant inequalities is not important but their combinatorial structure is.

Also in a discussion at the IBM Scientific Computing Symposium on Combinatorial Problems (March 1964 in Yorktown Heights, New York), Edmonds emphasized that the number of facets of a polyhedron is not a measure of the complexity of the associated optimization problem (see Gomory [1966]):

> I do not believe there is any reason for taking as a measure of the algorithmic difficulty of a class of combinatorial extremum problems the number of faces in the associated polyhedra. For example, consider the generalization of the assignment problem from bipartite graphs to arbitrary graphs. Unlike the case of bipartite graphs, the number of faces in the associated polyhedron increases exponentially with the size of the graph. On the other hand, there is an algorithm for this generalized assignment problem which has an upper bound on the work involved just as good as the upper bound for the bipartite assignment problem.

After having received support from H.W. Kuhn and referring to Kuhn's maximum-weight bipartite matching algorithm, Edmonds continued:

> This algorithm depends crucially on what amounts to knowing all the bounding inequalities of the associated convex polyhedron—and, as I said, there are many of them. The point is that the inequalities are known by an easily verifiable characterization rather than by an exhaustive listing—so their number is not important.
>
> This sort of thing should be expected for a class of extremum problems with a combinatorially special structure. For the traveling salesman problem, the vertices of the associated polyhedron have a simple characterization despite their number—so might the bounding inequalities have a simple characterization despite their number. At least we should hope they have, because finding a really good traveling salesman algorithm is undoubtedly equivalent to finding such a characterization.

So Edmonds was aware of the correlation of good algorithms and polyhedral characterizations, which later got further support by the ellipsoid method.

Also during the 1960s and 1970s, Fulkerson designed the clarifying framework of blocking and antiblocking polyhedra, throwing new light by the classical polarity of vertices and facets of polyhedra on combinatorial min-max relations and enabling, with a theorem of Lehman, the deduction of one polyhedral characterization from another. It stood at the basis of the solution of Berge's perfect graph conjecture in 1972 by Lovász, and it also inspired Seymour to obtain several other basic results in polyhedral combinatorics.

1.5b. Further notes

Raghavan and Thompson [1987] showed that randomized rounding of an optimum fractional solution to a combinatorial optimization problem yields, with high probability, an integer solution with objective value close to the value of the fractional solution (hence at least as close to the optimum value of the combinatorial problem). Related results were presented by Raghavan [1988], Plotkin, Shmoys, and Tardos [1991,1995], and Srinivasan [1995,1999].

Introductions to combinatorial optimization (and more than that) can be found in the books by Lawler [1976b], Papadimitriou and Steiglitz [1982], Sysło, Deo, and Kowalik [1983], Nemhauser and Wolsey [1988], Parker and Rardin [1988], Cook, Cunningham, Pulleyblank, and Schrijver [1998], Mehlhorn and Näher [1999], and Korte and Vygen [2000]. Focusing on applying geometric algorithms in combinatorial optimization are Lovász [1986] and Grötschel, Lovász, and Schrijver [1988]. Bibliographies on combinatorial optimization were given by Kastning [1976], Golden and Magnanti [1977], Hausmann [1978b], von Randow [1982,1985,1990], and O'hEigeartaigh, Lenstra, and Rinnooy Kan [1985].

Survey papers on polyhedral combinatorics and min-max relations were presented by Hoffman [1979], Pulleyblank [1983,1989], Schrijver [1983a,1986a,1987, 1995], and Grötschel [1985], on geometric methods in combinatorial optimization by Grötschel, Lovász, and Schrijver [1984b], and on polytopes and complexity by Papadimitriou [1984].