

Contents

Part I. Foundations of Noncommutative Geometry and Basic Model Building

Ralf Holtkamp

1	Spectral Triples and Abstract Yang-Mills Functional	4
1.1	Spectral Triples	4
1.2	Universal Differential Graded Algebra	5
1.3	Vector Potentials, Universal Connections	5
1.4	Quotient Differential Graded Algebra	6
1.5	Inner Product	7
1.6	Curvature and Yang-Mills Functional	8

Ralf Meyer

2	Real Spectral Triples and Charge Conjugation	11
2.1	Real Structures on Even Spectral Triples	11
2.2	Spin ^C Manifolds and Charge Conjugation	13
2.3	Real Structures via Clifford Algebras	15
2.4	Real Structures of Odd Dimension	17
2.5	Relations to Real K-Homology	18
2.6	Real Structures on the NC Torus	20

Michael Frank

3	The Commutative Case:	
	Spinors, Dirac Operator and de Rham Algebra	21
3.1	The Theorems by Gel'fand and Serre-Swan	21
3.2	Hermitian Structures and Frames for Sets of Sections	26
3.3	Clifford and Spinor Bundles, Spin Manifolds	28
3.4	Spin Connection and Dirac Operator	31
3.5	The Universal Differential Algebra $\Omega C^\infty(M)$ and Connes' Differential Algebra $\Omega_{\mathcal{D}} C^\infty(M)$	33
3.6	The Exterior Algebra Bundle $\Lambda(M)$ and the de Rham Complex	35
3.7	$\Omega_{\mathcal{D}} C^\infty(M)$ Versus $\Lambda(M)$	36

Peter M. Alberti and Reiner Matthes

4	Connes' Trace Formula and Dirac Realization of Maxwell and Yang-Mills Action	40
4.1	Generalities on Traces on C^* - and W^* -algebras	40
4.2	Examples of Traces	43
4.3	Examples of Singular Traces on $B(\mathcal{H})$	49
4.4	Calculating the Dixmier Trace	56
4.5	The Connes' Trace Theorem and its Application, Preliminaries	60
4.6	Connes' Trace Theorem	64
4.7	Classical Yang-Mills Actions	72

Bernd Ammann and Christian Bär

5	The Einstein-Hilbert Action as a Spectral Action	75
5.1	Generalized Laplacians and the Heat Equation	75
5.2	The Formal Heat Kernel	80
5.3	Dirac Operators and Weitzenböck Formulas	88
5.4	Integration and Dixmier Trace	91
5.5	Variational Formulas and the Einstein-Hilbert Action	93
5.6	Einstein-Hilbert Action and Wodzicki Residue	101

Ryszard Nest, Elmar Vogt, and Wend Werner

6	Spectral Action and the Connes-Chamseddine Model	109
6.1	The Spectral Action Principle	109
6.2	Example: Gravity Coupled to One Gauge Field	111
6.3	Asymptotic Expansion	113
6.4	First Example, Final Calculation	117
6.5	Gravity Coupled to the Standard Model	127

**Part II. The Lagrangian of the Standard Model
Derived from Noncommutative Geometry**

Harald Upmeyer

7	Dirac Operator and Real Structure on Euclidean and Minkowski Spacetime	136
7.1	γ -Matrices on Flat and Curved Spacetime	136
7.2	Levi-Civita Connection and Dirac Operator	144
7.3	Real Structure on Spacetime	147
7.4	Trace Formulas and Inner Products	150

Karen Elsner, Holger Neumann, and Harald Upmeyer

8	The Electro-weak Model	152
8.1	Noncommutative Matter Fields	152
8.2	Noncommutative Gauge Fields	155
8.3	Noncommutative Gauge Action Functional	165
8.4	Noncommutative Matter Action Functional	170

Karen Elsner, Holger Neumann, and Harald Upmeyer

9	The Full Standard Model	172
9.1	Noncommutative Matter Fields	172
9.2	Noncommutative Gauge Fields	179
9.3	Noncommutative Gauge Action Functional	206
9.4	Noncommutative Matter Action Functional	211

Holger Neumann and Harald Upmeyer

10	Standard Model Coupled with Gravity	216
10.1	Generalized Dirac Operators	216
10.2	Spectral Action and Heat Kernel Invariants	224

Florian Scheck

11	The Higgs Mechanism and Spontaneous Symmetry Breaking	230
11.1	Historical Note	230
11.2	Spontaneous Symmetry Breaking and Goldstone Theorem	232
11.3	Spontaneous Symmetry Breaking in Yang-Mills Theory	234
11.4	The Case of the Electroweak Model: Bosonic Sector	235
11.5	Electroweak Model: Adding Quarks and Leptons	238
11.6	Remarks About Fermionic Mass Generation	240

Part III. New Directions in Noncommutative Geometry and Mathematical Physics

Bruno Iochum

12	The Impact of NC Geometry in Particle Physics	244
12.1	Why Noncommutative Geometry?	244
12.2	Spectral Triples	245
12.3	Technical Points	247
12.4	The Noncommutative Highway	248
12.5	Computation of Higgs and W Masses	252
12.6	Parameter Counting	253
12.7	The Renormalization Machinery	255
12.8	Noncommutative Relativity	257
12.9	Conclusions	258

Rainer Häußling

13	The $su(2 1)$ Model of Electroweak Interactions and Its Connection to NC Geometry	260
13.1	Introduction and Motivation	260
13.2	The Bosonic Part of the Model	260
13.3	The Fermionic Part of the Model	267
13.4	The Connection to the Connes-Lott Model	269
13.5	Conclusions	270

Klaus Fredenhagen

14 Quantum Fields and Noncommutative Spacetime 271
 14.1 Noncommutative Spacetime and Uncertainty Relations 271
 14.2 Noncommutative Spacetime and Quantum Field Theory 273
 14.3 Interactions and Noncommutative Geometry 274
 14.4 Gauge Theories on Noncommutative Spacetime 276

Edwin Langmann

15 NC Geometry and Quantum Fields: Simple Examples 278
 15.1 Introduction 278
 15.2 Preliminaries 279
 15.3 Story I: Chern-Simons Terms from Effective Actions 284
 15.4 Story II: Regularization: Elementary Examples 286
 15.5 Story III: Regularized Traces of Operators 288
 15.6 Story IV: Yang-Mills Actions from Dirac Operators 294
 15.7 Final Remarks 297

Giovanny Landi

16 Dirac Eigenvalues as Dynamical Variables 299
 16.1 Introduction 299
 16.2 Noncommutative Geometry and Gravity 300
 16.3 From the Metric to the Eigenvalues 303
 16.4 Action and Field Equations 307
 16.5 Poisson Brackets for the Eigenvalues 309
 16.6 Final Remarks 311

Raimar Wulkenhaar

17 Hopf Algebras in Renormalization and NC Geometry 313
 17.1 Introductory Remarks 313
 17.2 The Hopf Algebra of Connes–Moscovici 313
 17.3 Rooted Trees 317
 17.4 Feynman Graphs and Rooted Trees 319
 17.5 A Toy Model: Iterated Integrals 321

Fedele Lizzi

18 NC Geometry of Strings and Duality Symmetry 325
 18.1 String Theory and T-duality 325
 18.2 Interacting Strings and Spectral Triples 328
 18.3 Compactification and Noncommutative Torus 333
 18.4 Noncommutative Configuration Space
 and Spectral Geometry 334
 18.5 Conclusions 337

References 338