

# Contents

<b>Preface to the Second Edition</b> .....	vii
<b>Preface to the First Edition</b> .....	ix
<b>1. Derivative Background</b> .....	1
1.1 Financial Markets and Instruments .....	2
1.1.1 Derivative Instruments .....	2
1.1.2 Underlying Securities .....	4
1.1.3 Markets .....	5
1.1.4 Types of Traders .....	6
1.1.5 Modeling Assumptions .....	6
1.2 Arbitrage .....	8
1.3 Arbitrage Relationships .....	11
1.3.1 Fundamental Determinants of Option Values .....	11
1.3.2 Arbitrage Bounds .....	13
1.4 Single-period Market Models .....	15
1.4.1 A Fundamental Example .....	15
1.4.2 A Single-period Model .....	18
1.4.3 A Few Financial-economic Considerations .....	25
Exercises .....	26
<b>2. Probability Background</b> .....	29
2.1 Measure .....	30
2.2 Integral .....	34
2.3 Probability .....	37
2.4 Equivalent Measures and Radon-Nikodým Derivatives .....	42
2.5 Conditional Expectation .....	44
2.6 Modes of Convergence .....	51
2.7 Convolution and Characteristic Functions .....	53
2.8 The Central Limit Theorem .....	57
2.9 Asset Return Distributions .....	61
2.10 Infinite Divisibility and the Lévy-Khintchine Formula .....	63
2.11 Elliptically Contoured Distributions .....	65
2.12 Hyperbolic Distributions .....	67

Exercises .....	71
<b>3. Stochastic Processes in Discrete Time .....</b>	<b>75</b>
3.1 Information and Filtrations .....	75
3.2 Discrete-parameter Stochastic Processes .....	77
3.3 Definition and Basic Properties of Martingales .....	78
3.4 Martingale Transforms .....	80
3.5 Stopping Times and Optional Stopping .....	82
3.6 The Snell Envelope and Optimal Stopping .....	88
3.7 Spaces of Martingales .....	94
3.8 Markov Chains .....	96
Exercises .....	98
<b>4. Mathematical Finance in Discrete Time .....</b>	<b>101</b>
4.1 The Model .....	101
4.2 Existence of Equivalent Martingale Measures .....	105
4.2.1 The No-arbitrage Condition .....	105
4.2.2 Risk-Neutral Pricing .....	112
4.3 Complete Markets: Uniqueness of EMMs .....	116
4.4 The Fundamental Theorem of Asset Pricing: Risk-Neutral Valuation .....	118
4.5 The Cox-Ross-Rubinstein Model .....	121
4.5.1 Model Structure .....	122
4.5.2 Risk-neutral Pricing .....	124
4.5.3 Hedging .....	126
4.6 Binomial Approximations .....	130
4.6.1 Model Structure .....	130
4.6.2 The Black-Scholes Option Pricing Formula .....	131
4.6.3 Further Limiting Models .....	136
4.7 American Options .....	138
4.7.1 Theory .....	138
4.7.2 American Options in the CRR Model .....	141
4.8 Further Contingent Claim Valuation in Discrete Time .....	143
4.8.1 Barrier Options .....	143
4.8.2 Lookback Options .....	144
4.8.3 A Three-period Example .....	145
4.9 Multifactor Models .....	147
4.9.1 Extended Binomial Model .....	147
4.9.2 Multinomial Models .....	148
Exercises .....	150

**5. Stochastic Processes in Continuous Time** . . . . . 153

5.1 Filtrations; Finite-dimensional Distributions . . . . . 153

5.2 Classes of Processes . . . . . 155

    5.2.1 Martingales . . . . . 155

    5.2.2 Gaussian Processes . . . . . 158

    5.2.3 Markov Processes . . . . . 158

    5.2.4 Diffusions . . . . . 159

5.3 Brownian Motion . . . . . 160

    5.3.1 Definition and Existence . . . . . 160

    5.3.2 Quadratic Variation of Brownian Motion . . . . . 167

    5.3.3 Properties of Brownian Motion . . . . . 171

    5.3.4 Brownian Motion in Stochastic Modeling . . . . . 173

5.4 Point Processes . . . . . 175

    5.4.1 Exponential Distribution . . . . . 175

    5.4.2 The Poisson Process . . . . . 176

    5.4.3 Compound Poisson Processes . . . . . 176

    5.4.4 Renewal Processes . . . . . 177

5.5 Lévy Processes . . . . . 179

    5.5.1 Distributions . . . . . 179

    5.5.2 Lévy Processes . . . . . 181

    5.5.3 Lévy Processes and the Lévy-Khintchine Formula . . . . . 183

5.6 Stochastic Integrals; Itô Calculus . . . . . 187

    5.6.1 Stochastic Integration . . . . . 187

    5.6.2 Itô's Lemma . . . . . 193

    5.6.3 Geometric Brownian Motion . . . . . 196

5.7 Stochastic Calculus for Black-Scholes Models . . . . . 198

5.8 Stochastic Differential Equations . . . . . 202

5.9 Likelihood Estimation for Diffusions . . . . . 206

5.10 Martingales, Local Martingales and Semi-martingales . . . . . 209

    5.10.1 Definitions . . . . . 209

    5.10.2 Semi-martingale Calculus . . . . . 211

    5.10.3 Stochastic Exponentials . . . . . 215

    5.10.4 Semi-martingale Characteristics . . . . . 217

5.11 Weak Convergence of Stochastic Processes . . . . . 219

    5.11.1 The Spaces  $C^d$  and  $D^d$  . . . . . 219

    5.11.2 Definition and Motivation . . . . . 220

    5.11.3 Basic Theorems of Weak Convergence . . . . . 222

    5.11.4 Weak Convergence Results for Stochastic Integrals . . . . . 223

Exercises . . . . . 225

**6. Mathematical Finance in Continuous Time** . . . . . 229

6.1 Continuous-time Financial Market Models . . . . . 229

    6.1.1 The Financial Market Model . . . . . 229

    6.1.2 Equivalent Martingale Measures . . . . . 232

    6.1.3 Risk-neutral Pricing . . . . . 235

6.1.4	Changes of Numéraire	239
6.2	The Generalized Black-Scholes Model	242
6.2.1	The Model	242
6.2.2	Pricing and Hedging Contingent Claims	250
6.2.3	The Greeks	254
6.2.4	Volatility	255
6.3	Further Contingent Claim Valuation	258
6.3.1	American Options	258
6.3.2	Asian Options	260
6.3.3	Barrier Options	263
6.3.4	Lookback Options	266
6.3.5	Binary Options	269
6.4	Discrete- versus Continuous-time Market Models	270
6.4.1	Discrete- to Continuous-time Convergence Reconsidered	270
6.4.2	Finite Market Approximations	271
6.4.3	Examples of Finite Market Approximations	274
6.4.4	Contiguity	280
6.5	Further Applications of the Risk-neutral Valuation Principle	281
6.5.1	Futures Markets	281
6.5.2	Currency Markets	285
	Exercises	287
<b>7.</b>	<b>Incomplete Markets</b>	<b>289</b>
7.1	Pricing in Incomplete Markets	289
7.1.1	A General Option-Pricing Formula	289
7.1.2	The Esscher Measure	292
7.2	Hedging in Incomplete Markets	295
7.2.1	Quadratic Principles	296
7.2.2	The Financial Market Model	297
7.2.3	Equivalent Martingale Measures	299
7.2.4	Hedging Contingent Claims	300
7.2.5	Mean-variance Hedging and the Minimal ELMM	305
7.2.6	Explicit Example	307
7.2.7	Quadratic Principles in Insurance	312
7.3	Stochastic Volatility Models	314
7.4	Models Driven by Lévy Processes	318
7.4.1	Introduction	318
7.4.2	General Lévy-process Based Financial Market Model	319
7.4.3	Existence of Equivalent Martingale Measures	321
7.4.4	Hyperbolic Models: The Hyperbolic Lévy Process	323

<b>8. Interest Rate Theory</b>	327
8.1 The Bond Market	328
8.1.1 The Term Structure of Interest Rates	328
8.1.2 Mathematical Modelling	330
8.1.3 Bond Pricing, ..	334
8.2 Short-rate Models	336
8.2.1 The Term-structure Equation	337
8.2.2 Martingale Modelling	338
8.2.3 Extensions: Multi-Factor Models	342
8.3 Heath-Jarrow-Morton Methodology	343
8.3.1 The Heath-Jarrow-Morton Model Class	343
8.3.2 Forward Risk-neutral Martingale Measures	346
8.3.3 Completeness	348
8.4 Pricing and Hedging Contingent Claims	350
8.4.1 Short-rate Models	350
8.4.2 Gaussian HJM Framework	351
8.4.3 Swaps	353
8.4.4 Caps	354
8.5 Market Models of LIBOR- and Swap-rates	356
8.5.1 Description of the Economy	356
8.5.2 LIBOR Dynamics Under the Forward LIBOR Measure	357
8.5.3 The Spot LIBOR Measure	361
8.5.4 Valuation of Caplets and Floorlets in the LMM	362
8.5.5 The Swap Market Model	363
8.5.6 The Relation Between LIBOR- and Swap-market Models	367
8.6 Potential Models and the Flesaker-Hughston Framework	368
8.6.1 Pricing Kernels and Potentials	368
8.6.2 The Flesaker-Hughston Framework	370
Exercises	372
<b>9. Credit Risk</b>	375
9.1 Aspects of Credit Risk	376
9.1.1 The Market	376
9.1.2 What Is Credit Risk?	376
9.1.3 Portfolio Risk Models	377
9.2 Basic Credit Risk Modeling	378
9.3 Structural Models	379
9.3.1 Merton's Model	379
9.3.2 A Jump-diffusion Model	382
9.3.3 Structural Model with Premature Default	384
9.3.4 Structural Model with Stochastic Interest Rates	388
9.3.5 Optimal Capital Structure – Leland's Approach	389
9.4 Reduced Form Models	390

9.4.1	Intensity-based Valuation of a Defaultable Claim . . . . .	390
9.4.2	Rating-based Models . . . . .	397
9.5	Credit Derivatives . . . . .	399
9.6	Portfolio Credit Risk Models . . . . .	400
9.7	Collateralized Debt Obligations (CDOs) . . . . .	404
9.7.1	Introduction . . . . .	404
9.7.2	Review of Modelling Methods . . . . .	405
<b>A.</b>	<b>Hilbert Space</b> . . . . .	<b>409</b>
<b>B.</b>	<b>Projections and Conditional Expectations</b> . . . . .	<b>411</b>
<b>C.</b>	<b>The Separating Hyperplane Theorem</b> . . . . .	<b>415</b>
	<b>Bibliography</b> . . . . .	<b>417</b>
	<b>Index</b> . . . . .	<b>432</b>