Preface

Cyclic homology was introduced in the early eighties independently by Connes and Tsygan. They came from different directions. Connes wanted to associate homological invariants to K-homology classes and to describe the index pairing with K-theory in that way, while Tsygan was motivated by algebraic K-theory and Lie algebra cohomology. At the same time Karoubi had done work on characteristic classes that led him to study related structures, without however arriving at cyclic homology properly speaking.

Many of the principal properties of cyclic homology were already developed in the fundamental article of Connes and in the long paper by Feigin–Tsygan. In the sequel, cyclic homology was recognized quickly by many specialists as a new intriguing structure in homological algebra, with unusual features. In a first phase it was tried to treat this structure as well as possible within the traditional framework of homological algebra. The cyclic homology groups were computed in many examples and new important properties such as product structures, excision for H-unital ideals, or connections with cyclic objects and simplicial topology, were established. An excellent account of the state of the theory after that phase is given in the book of Loday.

This book is an attempt at an account of the present state of cyclic theory that covers in particular a number of more recent results that have changed the face of the theory to a certain extent. An essential feature of cyclic homology which is not so well captured by the ordinary graded cyclic homology groups HC_n is the S-operator, which reflects in fact a filtration – rather than a grading – on cyclic homology, along with the associated periodic cyclic theory introduced by Connes. The most characteristic properties of cyclic homology hold only after stabilization by the S-operator. Important steps beyond the classical graded framework were taken by Connes when he introduced the so-called entire theory, and by Goodwillie in his result on nilpotent extensions. The spirit here is to work not with the individual cyclic homology groups in finite dimension, but with the infinite cyclic bicomplex. This global approach also turned out to be the most appropriate for various index theorems. It was put on a new basis by Cuntz–Quillen who used free (or more generally quasifree) resolutions of an algebra and a non-commutative de Rham complex, to describe all the cyclic-type invariants associated with that algebra. This also led to proofs of excision for general extensions in periodic, entire or other cyclic theories. Excision in its bivariant form in turn immediately leads to a natural multiplicative transformation – the bivariant Chern–Connes character – from bivariant K-theory, provided that this can be defined, to bivariant cyclic homology. This general transformation contains most previously constructed characters from K-theoretic invariants to cyclic theory invariants. The formalism using the non-commutative de Rham complex and free resolutions of an algebra is also the natural basis for cyclic homology theories generalizing the entire theory that can be applied to a wide class of topological algebras, including Banach- and C^*-algebras.

Other new techniques in cyclic homology theory were developed by Nest and Tsygan who systematically treat cyclic chains as noncommutative differential forms and Hochschild cochains as noncommutative multivector fields. In this way they recover the standard algebra arising in calculus to a surprisingly large extent. The resulting algebraic structures on the Hochschild and cyclic complexes are applied to prove far-reaching generalizations of the Atiyah–Singer index theorem culminating in the theorem of Bressler–Nest–Tsygan. The algebraic structures were further enriched in the work of Tamarkin and Tamarkin–Tsygan and used to reprove and extend Kontsevich's results on classification of deformation quantizations.

On the other hand, from the beginning, Connes developed cyclic theory into a powerful tool in non-commutative geometry. Striking applications of cyclic homology to geometry appeared already in his work on the transverse fundamental class for actions of discrete groups on a manifold and for transversally oriented foliations, including an interpretation of the Godbillon–Vey class in cyclic cohomology. Another such application is the proof of the Novikov conjecture for hyperbolic groups by Connes–Moscovici using cyclic cohomology and a 'higher' index theorem. More recently, Connes–Moscovici proved two other important index theorems which required new techniques in cyclic homology. The first one is a general abstract local index formula for spectral triples, which works for general algebras sharing some of the characteristic features of the algebra of pseudodifferential operators, and applies to many natural operators in non-commutative geometry. Locality is reflected by the fact that one uses only the Dixmier–Wodzicki trace for the description of the cyclic cocycles determining the index. The application of their local index formula to tranversally hypoelliptic operators on foliations led Connes and Moscovici to the introduction of a new type of cyclic homology for Hopf algebras. The total index in transverse geometry is expressed in terms of a characteristic map from the cyclic homology of a Hopf algebra playing the role of a symmetry group in this situation. The contribution by G. Skandalis in this volume is devoted to an outline of these last two theorems. A much more complete account is planned for another volume in this series.

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