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# Contents

|  |     |
|--|-----|
| <b>Cyclic Theory, Bivariant <math>K</math>-Theory and the Bivariant Chern-Connes Character</b><br><i>Joachim Cuntz</i> .....   | 1   |
| <b>Cyclic Homology</b><br><i>Boris Tsygan</i> .....  | 73  |
| <b>Noncommutative Geometry, the Transverse Signature Operator, and Hopf Algebras</b><br>[after A. Connes and H. Moscovici]<br><i>Georges Skandalis (translated by Raphaël Ponge and Nick Wright)</i> ..... | 115 |
| <b>Index</b> .....   | 135 |

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# Cyclic Theory, Bivariant $K$ -Theory and the Bivariant Chern-Connes Character\*

Joachim Cuntz

|          |   |    |
|----------|---|----|
| <b>1</b> | <b>Introduction</b> .....                               | 2  |
| <b>2</b> | <b>Cyclic Theory</b> .....                              | 5  |
| 2.1      | Preliminaries .....                                     | 5  |
| 2.2      | Cyclic homology via the cyclic bicomplex .....          | 9  |
| 2.3      | Operators on differential forms .....                   | 13 |
| 2.4      | The periodic theory .....                               | 16 |
| 2.5      | Cyclic homology via the $X$ -complex .....              | 21 |
| 2.6      | Cyclic homology as non-commutative de Rham theory ..... | 27 |
| 2.7      | Homotopy invariance for cyclic theory .....             | 29 |
| 2.8      | Morita invariance .....                                 | 31 |
| 2.9      | Excision .....  | 33 |
| 2.10     | Chern character for $K$ -theory elements .....          | 34 |
| <b>3</b> | <b>Cyclic Theory for locally convex algebras</b> .....  | 36 |
| 3.1      | General modifications .....                             | 36 |
| 3.2      | De Rham theory for differentiable manifolds .....       | 38 |
| 3.3      | Cyclic homology for Schatten ideals .....               | 38 |
| 3.4      | Cyclic cocycles associated with Fredholm modules .....  | 40 |
| <b>4</b> | <b>Bivariant <math>K</math>-Theory</b> .....            | 42 |
| 4.1      | Bivariant $K$ -theory for locally convex algebras ..... | 43 |
| 4.2      | The bivariant Chern-Connes character .....              | 48 |
| <b>5</b> | <b>Infinite-dimensional Cyclic Theories</b> .....       | 51 |
| 5.1      | Entire cyclic cohomology .....                          | 51 |
| 5.2      | Local cyclic cohomology .....                           | 57 |

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|          |  |    |
|----------|--|----|
| <b>A</b> | <b>Locally convex algebras</b> .....           | 61 |
| A.1      | Algebras of differentiable functions .....     | 62 |
| A.2      | The smooth tensor algebra .....                | 62 |
| A.3      | The free product of two $m$ -algebras .....    | 63 |
| A.4      | The algebra of smooth compact operators .....  | 64 |
| A.5      | The Schatten ideals $\ell^p(H)$ .....          | 65 |
| A.6      | The smooth Toeplitz algebra .....              | 65 |
| <b>B</b> | <b>Standard extensions</b> .....               | 66 |
| B.1      | The suspension extension .....                 | 66 |
| B.2      | The free extension .....                       | 67 |
| B.3      | The universal two-fold trivial extension ..... | 67 |
| B.4      | The Toeplitz extension .....                   | 68 |
|          | <b>References</b> .....                        | 69 |

## 1 Introduction

The two fundamental “machines” of non-commutative geometry are (bivariant) topological  $K$ -theory and cyclic homology. In the present contribution we describe these two theories and their connections. Cyclic theory can be viewed as a far reaching generalization of the classical de Rham cohomology, while bivariant  $K$ -theory includes the topological  $K$ -theory of Atiyah-Hirzebruch as a special case.

The classical commutative theories can be extended to a degree of generality which is quite striking. It is important to note however that this extension is by no means simply based on generalizations of the existing classical methods. The constructions are quite different and give, in the commutative case, a new approach and an unexpected interpretation of the well-known classical theories. One aspect is that some of the properties of the two theories become visible only in the non-commutative category. For instance, both theories have certain universality properties in this setting.

Bivariant  $K$ -theory has first been defined and developed by Kasparov on the category of  $C^*$ -algebras (possibly with the action of a locally compact group) thereby unifying and decisively extending previous work by Atiyah-Hirzebruch, Brown-Douglas-Fillmore and others. Kasparov also applied his bivariant theory to obtain striking positive results on the Novikov conjecture. Very recently [13], it was discovered that in fact, bivariant topological  $K$ -theories can be defined on a wide variety of topological algebras ranging from rather general locally convex algebras to e.g. Banach algebras or  $C^*$ -algebras (in fact, even algebras without a specified topology can be covered to some extent). If  $E$  is the covariant functor from such a category  $C$  of algebras given by topological  $K$ -theory or also by periodic cyclic homology, then it possesses the following three fundamental properties:

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# Cyclic Homology<sup>\*</sup>

Boris Tsygan

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|----------|--|----|
| <b>1</b> | <b>Introduction</b>                                      | 74 |
| <b>2</b> | <b>Hochschild and cyclic homology of algebras</b>        | 76 |
| 2.1      | Homology of differential graded algebras                 | 78 |
| 2.2      | The Hochschild cochain complex                           | 79 |
| 2.3      | Products on Hochschild and cyclic complexes              | 80 |
| 2.4      | Pairings between chains and cochains                     | 82 |
| 2.5      | $A_\infty$ structure on $C_\bullet(C^\bullet(A))$        | 84 |
| <b>3</b> | <b>The cyclic complex <math>C_\bullet^\lambda</math></b> | 86 |
| 3.1      | Definition   | 86 |
| 3.2      | The reduced cyclic complex                               | 87 |
| 3.3      | Relation to Lie algebra homology                         | 88 |
| 3.4      | The connecting morphism                                  | 88 |
| 3.5      | Other operations on the cyclic complexes                 | 90 |
| 3.6      | Rigidity of periodic cyclic homology                     | 91 |
| <b>4</b> | <b>Noncommutative differential calculus</b>              | 92 |
| 4.1      | Gerstenhaber algebras                                    | 92 |
| 4.2      | The Gerstenhaber algebra $\mathcal{V}^\bullet(A)$        | 92 |
| 4.3      | Calculi  | 93 |
| 4.4      | The calculus $\text{Calc}(A)$                            | 94 |
| 4.5      | Enveloping algebra of a Gerstenhaber algebra             | 95 |
| 4.6      | Relation to the $A_\infty$ structure on chains           | 95 |
| <b>5</b> | <b>Cyclic objects</b>                                    | 96 |
| 5.1      | Simplicial objects                                       | 96 |
| 5.2      | Cyclic objects   | 96 |

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|          |   |     |
|----------|---|-----|
| <b>6</b> | <b>Examples</b> .....   | 98  |
| 6.1      | Smooth functions .....  | 98  |
| 6.2      | Commutative algebras .....                                    | 99  |
| 6.3      | Rings of differential operators .....                         | 100 |
| 6.4      | Rings of complete symbols .....                               | 101 |
| 6.5      | Rings of pseudodifferential operators .....                   | 101 |
| 6.6      | Noncommutative tori .....                                     | 102 |
| 6.7      | Deformation quantization .....                                | 102 |
| 6.8      | The Brylinski spectral sequence .....                         | 105 |
| 6.9      | Deformation quantization: general case .....                  | 105 |
| 6.10     | Group rings .....   | 106 |
| <b>7</b> | <b>Index theorems</b> .....                                   | 107 |
| 7.1      | Index theorem for deformations of symplectic structures ..... | 107 |
| 7.2      | Index theorem for holomorphic symplectic deformations .....   | 108 |
| 7.3      | General index theorem for deformations .....                  | 108 |
| <b>8</b> | <b>Riemann-Roch theorem for D-modules</b> .....               | 109 |
|          | <b>References</b> .....                                       | 110 |

## 1 Introduction

Many geometric objects associated to a manifold  $M$  can be expressed in terms of an appropriate algebra  $A$  of functions on  $M$  (measurable, continuous, smooth, holomorphic, algebraic, ...). Very often those objects can be defined in a way that is applicable to any algebra  $A$ , commutative or not. Study of associative algebras by means of such objects of geometric origin is the subject of noncommutative geometry [12, 48]. The Hochschild and cyclic (co)homology theory is the part of noncommutative geometry which generalizes the classical differential and integral calculus. The geometric objects being generalized to the noncommutative setting are differential forms, densities, multivector fields, etc.

In our exposition, the primary object is the negative cyclic complex  $CC_{\bullet}^{-}(A)$ . Other complexes, namely the Hochschild chain complex  $C_{\bullet}(A)$ , the periodic cyclic complex  $CC_{\bullet}^{\text{per}}(A)$ , and the cyclic complex  $CC_{\bullet}(A)$ , are defined as results of some natural procedure applied to  $CC_{\bullet}^{-}(A)$ . The cyclic homology is the homology of the cyclic complex  $CC_{\bullet}(A)$ . It was originally defined using another standard complex which we denote by  $C_{\bullet}^{\lambda}(A)$ . The study of this latter complex has a distinctly different flavor, mainly coming from the fact that it is related to the Lie algebra homology.

The above complexes are noncommutative versions of the space of differential forms (the Hochschild chain complex) and of the De Rham complex. One also defines the Hochschild cochain complex  $C^{\bullet}(A, A)$  which is a noncommutative analogue of the space of multivector fields.

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# Noncommutative Geometry, the Transverse Signature Operator, and Hopf Algebras [after A. Connes and H. Moscovici]\*

Georges Skandalis (translated by Raphaël Ponge and Nick Wright)

|          |   |     |
|----------|---|-----|
| <b>1</b> | <b>Preliminaries</b> . . . . .  | 117 |
| 1.1      | Complex powers of pseudodifferential operators;<br>the Wodzicki-Guillemin residue . . . . . | 117 |
| 1.2      | Cyclic cohomology . . . . .   | 117 |
| 1.3      | $K$ -homology; $p$ -summable cycles . . . . .   | 118 |
| <b>2</b> | <b>The Local Index Formula</b> . . . . .  | 120 |
| <b>3</b> | <b>The Diff-Invariant Signature Operator</b> . . . . .                                      | 122 |
| 3.1      | The Hilbert space $\mathcal{H}$ and the algebra $\mathcal{A}$ . . . . .                     | 123 |
| 3.2      | The vertical part of the signature operator . . . . .                                       | 124 |
| 3.3      | The horizontal part of the signature operator . . . . .                                     | 124 |
| 3.4      | The spectral triple . . . . .   | 124 |
| <b>4</b> | <b>The “Transverse” Hopf Algebra</b> . . . . .  | 125 |
| 4.1      | Matched pairs of groups . . . . .   | 125 |
| 4.2      | The Hopf algebra $\mathcal{H}_n$ . . . . .  | 127 |
| 4.3      | Cyclic cohomology of Hopf algebras . . . . .  | 128 |
| 4.4      | The classifying map . . . . .   | 129 |
| 4.5      | Cyclic cohomology of the Hopf algebra $\mathcal{H}_n$ . . . . .                             | 129 |
| 4.6      | The cyclic cocycle $\Phi$ . . . . .   | 130 |
| 4.7      | Calculation of $\Phi$ . . . . .   | 131 |
| 4.8      | A variant of the Hopf algebra . . . . .   | 132 |
|          | <b>References</b> . . . . .   | 132 |

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