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Cyclic Theory, Bivariant *K*-Theory and the Bivariant Chern-Connes Character*

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1 Introduction

The two fundamental "machines" of non-commutative geometry are (bivariant) topological K-theory and cyclic homology. In the present contribution we describe these two theories and their connections. Cyclic theory can be viewed as a far reaching generalization of the classical de Rham cohomology, while bivariant K-theory includes the topological K-theory of Atiyah-Hirzebruch as a special case.

The classical commutative theories can be extended to a degree of generality which is quite striking. It is important to note however that this extension is by no means simply based on generalizations of the existing classical methods. The constructions are quite different and give, in the commutative case, a new approach and an unexpected interpretation of the well-known classical theories. One aspect is that some of the properties of the two theories become visible only in the non-commutative category. For instance, both theories have certain universality properties in this setting.

Bivariant K-theory has first been defined and developed by Kasparov on the category of C^*-algebras (possibly with the action of a locally compact group) thereby unifying and decisively extending previous work by Atiyah-Hirzebruch, Brown-Douglas-Fillmore and others. Kasparov also applied his bivariant theory to obtain striking positive results on the Novikov conjecture. Very recently [13], it was discovered that in fact, bivariant topological K-theories can be defined on a wide variety of topological algebras ranging from rather general locally convex algebras to e.g. Banach algebras or C^*-algebras (in fact, even algebras without a specified topology can be covered to some extent). If E is the covariant functor from such a category C of algebras given by topological K-theory or also by periodic cyclic homology, then it possesses the following three fundamental properties:

Cyclic Homology*

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1 Introduction

Many geometric objects associated to a manifold M can be expressed in terms of an appropriate algebra A of functions on M (measurable, continuous, smooth, holomorphic, algebraic, ...). Very often those objects can be defined in a way that is applicable to any algebra A, commutative or not. Study of associative algebras by means of such objects of geometric origin is the subject of noncommutative geometry [12, 48]. The Hochschild and cyclic (co)homology theory is the part of noncommutative geometry which generalizes the classical differential and integral calculus. The geometric objects being generalized to the noncommutative setting are differential forms, densities, multivector fields, etc.

In our exposition, the primary object is the negative cyclic complex $CC_\bullet^-(A)$. Other complexes, namely the Hochschild chain complex $C_\bullet(A)$, the periodic cyclic complex $CC_\bullet^{\mathrm{per}}(A)$, and the cyclic complex $CC_\bullet(A)$, are defined as results of some natural procedure applied to $CC_\bullet^-(A)$. The cyclic homology is the homology of the cyclic complex $CC_\bullet(A)$. It was originally defined using another standard complex which we denote by $C_\bullet^\lambda(A)$. The study of this latter complex has a distinctly different flavor, mainly coming from the fact that it is related to the Lie algebra homology.

The above complexes are noncommutative versions of the space of differential forms (the Hochschild chain complex) and of the De Rham complex. One also defines the Hochschild cochain complex $C^\bullet(A, A)$ which is a noncommutative analogue of the space of multivector fields.

Noncommutative Geometry, the Transverse Signature Operator, and Hopf Algebras [after A. Connes and H. Moscovici]*

Georges Skandalis (translated by Raphaël Ponge and Nick Wright)

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