
Contents

The Objective Method: Probabilistic Combinatorial Optimization and Local Weak Convergence <i>David Aldous, J. Michael Steele</i>	1
The Random-Cluster Model <i>Geoffrey Grimmett</i>	73
Models of First-Passage Percolation <i>C. Douglas Howard</i>	125
Relaxation Times of Markov Chains in Statistical Mechanics and Combinatorial Structures <i>Fabio Martinelli</i>	175
Random Walks on Finite Groups <i>Laurent Saloff-Coste</i>	263
Index	347

The Objective Method: Probabilistic Combinatorial Optimization and Local Weak Convergence

David Aldous and J. Michael Steele

1	Introduction	2
1.1	A Motivating Example: the Assignment Problem	3
1.2	A Stalking Horse: the Partial Matching Problem	4
1.3	Organization of the Survey	4
2	Geometric Graphs and Local Weak Convergence	6
2.1	Geometric Graphs	6
2.2	\mathcal{G}_* as a Metric Space	7
2.3	Local Weak Convergence	8
2.4	The Standard Construction	8
2.5	A Prototype: The Limit of Uniform Random Trees	9
3	Maximal Weight Partial Matching on Random Trees	12
3.1	Weighted Matchings of Graphs in General	12
3.2	Our Case: Random Trees with Random Edge Weights	12
3.3	Two Obvious Guesses: One Right, One Wrong	12
3.4	Not Your Grandfather's Recursion	13
3.5	A Direct and Intuitive Plan	14
3.6	Characterization of the Limit of $B(T_n^{small})$	16
3.7	Characterization of the Limit of $B(T_n^{big})$	19
3.8	The Limit Theorem for Maximum Weight Partial Matchings	21
3.9	Closing the Loop: Another Probabilistic Solution of a Fixed-Point Equation	24
3.10	From Coupling to Stability – Thence to Convergence	26
3.11	Looking Back: Perspective on a Case Study	28
4	The Mean-Field Model of Distance	29
4.1	From Poisson Points in \mathbb{R}^d to a Simple Distance Model	29
4.2	The Poisson Weighted Infinite Tree – or, the PWIT	31
4.3	The Cut-off Components of a Weighted Graph and a PWIT	32

4.4	The Minimum Spanning Forests of an Infinite Graph	33
4.5	The Average Length Per Vertex of the MSF of a PWIT	34
4.6	The Connection to Frieze’s $\zeta(3)$ Theorem	35
5	Minimal Cost Perfect Matchings	37
5.1	A Natural Heuristic – Which Fails for a Good Reason	38
5.2	Involution Invariance and the Standard Construction	39
5.3	Involution Invariance and the Convergence of MSTs	42
5.4	A Heuristic That Works by Focusing on the Unknown	46
5.5	A Distributional Identity with a Logistic Solution	47
5.6	A Stochastic Process that Constructs a Matching	49
5.7	Calculation of a Limiting Constant: $\pi^2/6$	52
5.8	Passage from a PWIT Matching to a K_n Matching	53
5.9	Finally – Living Beyond One’s Means	55
6	Problems in Euclidean Space	56
6.1	A Motivating Problem	57
6.2	Far Away Places and Their Influence	59
6.3	Euclidean Methods and Some Observations in Passing	62
6.4	Recurrence of Random Walks in Limits of Planar Graphs	65
7	Limitations, Challenges, and Perspectives	66
	References	69

1 Introduction

This survey describes a general approach to a class of problems that arise in combinatorial probability and combinatorial optimization. Formally, the method is part of weak convergence theory, but in concrete problems the method has a flavor of its own. A characteristic element of the method is that it often calls for one to introduce a new, infinite, probabilistic *object* whose *local properties* inform us about the *limiting properties* of a sequence of finite problems.

The name *objective method* hopes to underscore the value of shifting ones attention to the new, large random object with fixed distributional properties and way from the sequence of objects with changing distributions. The new object always provides us with some new information on the asymptotic behavior of the original sequence, and, in the happiest cases, the constants associated with the infinite object even permit us to find the elusive limit constants for that sequence.

The Random-Cluster Model

Geoffrey Grimmett

Abstract. The class of random-cluster models is a unification of a variety of stochastic processes of significance for probability and statistical physics, including percolation, Ising, and Potts models; in addition, their study has impact on the theory of certain random combinatorial structures, and of electrical networks. Much (but not all) of the physical theory of Ising/Potts models is best implemented in the context of the random-cluster representation. This systematic summary of random-cluster models includes accounts of the fundamental methods and inequalities, the uniqueness and specification of infinite-volume measures, the existence and nature of the phase transition, and the structure of the subcritical and supercritical phases. The theory for two-dimensional lattices is better developed than for three and more dimensions. There is a rich collection of open problems, including some of substantial significance for the general area of disordered systems, and these are highlighted when encountered. Amongst the major open questions, there is the problem of ascertaining the exact nature of the phase transition for general values of the cluster-weighting factor q , and the problem of proving that the critical random-cluster model in two dimensions, with $1 \leq q \leq 4$, converges when re-scaled to a stochastic Löwner evolution (SLE). Overall the emphasis is upon the random-cluster model for its own sake, rather than upon its applications to Ising and Potts systems.

1	Introduction	74
2	Potts and random-cluster processes	77
2.1	Random-cluster measures	77
2.2	Ising and Potts models	78
2.3	Random-cluster and Ising–Potts coupled	80
2.4	The limit as $q \downarrow 0$	82
2.5	Rank-generating functions	83
3	Infinite-volume random-cluster measures	84
3.1	Stochastic ordering	84
3.2	A differential formula	85

3.3	Conditional probabilities	85
3.4	Infinite-volume weak limits	86
3.5	Random-cluster measures on infinite graphs	88
3.6	The case $q < 1$	89
4	Phase transition, the big picture	91
4.1	Infinite open clusters	91
4.2	First- and second-order phase transition	92
5	General results in $d (\geq 2)$ dimensions	95
5.1	The subcritical phase, $p < p_c(q)$	95
5.2	The supercritical phase, $p > p_c(q)$	96
5.3	Near the critical point, $p \simeq p_c(q)$	98
6	In two dimensions	101
6.1	Graphical duality	101
6.2	Value of the critical point	103
6.3	First-order phase transition	104
6.4	SLE limit when $q \leq 4$	105
7	On complete graphs and trees	108
7.1	On complete graphs	108
7.2	On trees and non-amenable graphs	110
8	Time-evolutions of random-cluster models	111
8.1	Reversible dynamics	111
8.2	Coupling from the past	113
8.3	Swendsen–Wang dynamics	114
	References	116

1 Introduction

During a classical period, probabilists studied the behaviour of *independent* random variables. The emergent theory is rich, and is linked through theory and application to areas of pure/applied mathematics and to other sciences. It is however unable to answer important questions from a variety of sources concerning large families of *dependent* random variables. Dependence comes in many forms, and one of the targets of modern probability theory has been to derive robust techniques for studying it. The voice of statistical physics has been especially loud in the call for rigour in this general area. In a typical scenario, we are provided with an infinity of random variables, indexed by the vertices of some graph such as the cubic lattice, and which have some dependence structure governed by the geometry of the graph. Thus mathematicians and physicists have had further cause to relate probability and geometry. One

Models of First-Passage Percolation

C. Douglas Howard*

1	Introduction	126
1.1	The Basic Model and Some Fundamental Questions	126
1.2	Notation	128
2	The Time Constant	129
2.1	The Fundamental Processes of Hammersley and Welsh	129
2.2	About μ	131
2.3	Minimizing Paths	133
3	Asymptotic Shape and Shape Fluctuations	134
3.1	Shape Theorems for Standard FPP	134
3.2	About the Asymptotic Shape for Lattice FPP	138
3.3	FPP Based on Poisson Point Processes	140
3.4	Upper Bounds on Shape Fluctuations	143
3.5	Some Related Longitudinal Fluctuation Exponents	150
3.6	Monotonicity	151
4	Transversal Fluctuations and the Divergence of Shape Fluctuations	154
4.1	Transversal Fluctuation Exponents	154
4.2	Upper Bounds on ξ	155
4.3	Lower Bounds on χ	157
4.4	Lower Bounds on ξ	158
4.5	Fluctuations for Other Related Models	160
5	Infinite Geodesics and Spanning Trees	161
5.1	Semi-Infinite Geodesics and Spanning Trees	161
5.2	Coalescence and Another Spanning Tree in 2 Dimensions	165
5.3	Doubly-Infinite Geodesics	167
6	Summary of Some Open Problems	168
	References	170

* Research supported by NSF Grant DMS-02-03943.

Relaxation Times of Markov Chains in Statistical Mechanics and Combinatorial Structures

Fabio Martinelli

Abstract. In Markov chain Monte Carlo theory a particular Markov chain is run for a very long time until its distribution is close enough to the equilibrium measure. In recent years, for models of statistical mechanics and of theoretical computer science, there has been a flourishing of new mathematical ideas and techniques to rigorously control the time it takes for the chain to equilibrate. This has provided a fruitful interaction between the two fields and the purpose of this paper is to provide a comprehensive review of the state of the art.

1	Introduction	177
2	Mixing times for reversible, continuous-time Markov chains	180
2.1	Analytic methods	182
2.2	Tensorization of the Poincaré and logarithmic Sobolev inequalities	186
2.3	Geometric tools	188
2.4	Comparison methods	190
2.5	Coupling methods and block dynamics	192
3	Statistical mechanics models in \mathbb{Z}^d	194
3.1	Notation	194
3.2	Grand canonical Gibbs measures	195
3.3	Mixing conditions and absence of long-range order	197
3.4	Canonical Gibbs measures for lattice gases	201
3.5	The ferromagnetic Ising and Potts models	202
3.6	FK representation of Potts models	202
3.7	Antiferromagnetic models on an arbitrary graph: Potts and hard-core models	204
3.8	Model with random interactions	206
3.9	Unbounded spin systems	207

3.10	Ground states of certain quantum Heisenberg models as classical Gibbs measures	208
4	Glauber dynamics in \mathbb{Z}^d	211
4.1	The dynamics in a finite volume	211
4.2	The dynamics in an infinite volume	213
4.3	Graphical construction	214
4.4	Uniform ergodicity and logarithmic Sobolev constant	215
5	Mixing property versus logarithmic Sobolev constant in \mathbb{Z}^d	218
5.1	The auxiliary chain and sweeping out relations method	219
5.2	The renormalization group approach	220
5.3	The martingale method	222
5.4	The recursive analysis	225
5.5	Rapid mixing for unbounded spin systems	226
6	Torpid mixing in the phase coexistence region	227
6.1	Torpid mixing for the Ising model in $\Lambda \subset \mathbb{Z}^d$ with free boundary conditions	227
6.2	Interface driven mixing inside one phase	229
6.3	Torpid mixing for Potts model in \mathbb{Z}^d	231
7	Glauber dynamics for certain random systems in \mathbb{Z}^d	231
7.1	Combination of torpid and rapid mixing: the dilute Ising model	231
7.2	Relaxation to equilibrium for spin glasses	233
8	Glauber dynamics for more general structures	234
8.1	Glauber dynamics on trees and hyperbolic graphs	235
8.2	Glauber dynamics for the hard-core model	236
8.3	Cluster algorithms: the Swendsen–Wang dynamics for Potts models	237
9	Mixing time for conservative dynamics	238
9.1	Random transposition, Bernoulli–Laplace and symmetric simple exclusion	239
9.2	The asymmetric simple exclusion	240
9.3	The Kac model for the Boltzmann equation	245
9.4	Adsorbing staircase walks	247
10	Kawasaki dynamics for lattice gases	248
10.1	Diffusive scaling of the mixing time in the one-phase region	249
10.2	Torpid mixing in the phase coexistence region	252
	References	253

Random Walks on Finite Groups

Laurent Saloff-Coste*

Summary. Markov chains on finite sets are used in a great variety of situations to approximate, understand and sample from their limit distribution. A familiar example is provided by card shuffling methods. From this viewpoint, one is interested in the “mixing time” of the chain, that is, the time at which the chain gives a good approximation of the limit distribution. A remarkable phenomenon known as the cut-off phenomenon asserts that this often happens abruptly so that it really makes sense to talk about “the mixing time”. Random walks on finite groups generalize card shuffling models by replacing the symmetric group by other finite groups. One then would like to understand how the structure of a particular class of groups relates to the mixing time of natural random walks on those groups. It turns out that this is an extremely rich problem which is very far to be understood. Techniques from a great variety of different fields – Probability, Algebra, Representation Theory, Functional Analysis, Geometry, Combinatorics – have been used to attack special instances of this problem. This article gives a general overview of this area of research.

1	Introduction	264
2	Background and Notation	267
2.1	Finite Markov Chains	267
2.2	Invariant Markov Chains on Finite Groups	270
3	Shuffling Cards and the Cut-off Phenomenon	272
3.1	Three Examples of Card Shuffling	272
3.2	Exact Computations	274
3.3	The Cut-off Phenomenon	277
4	Probabilistic Methods	281
4.1	Coupling	281
4.2	Strong Stationary Times	285
5	Spectrum and Singular Values	289

* Research supported in part by NSF grant DMS 0102126

5.1	General Finite Markov Chains	289
5.2	The Random Walk Case	292
5.3	Lower Bounds	293
6	Eigenvalue Bounds Using Paths	296
6.1	Cayley Graphs	296
6.2	The Second Largest Eigenvalue	297
6.3	The Lowest Eigenvalue	300
6.4	Diameter Bounds, Isoperimetry and Expanders	302
7	Results Involving Volume Growth Conditions	308
7.1	Moderate Growth	308
7.2	Nilpotent Groups	311
7.3	Nilpotent Groups with many Generators	312
8	Representation Theory for Finite Groups	315
8.1	The General Set-up	315
8.2	Abelian Examples	317
8.3	Random Random Walks	323
9	Central Measures and Bi-invariant Walks	325
9.1	Characters and Bi-invariance	325
9.2	Random Transposition on the Symmetric Group	326
9.3	Walks Based on Conjugacy Classes of the Symmetric Group	328
9.4	Finite Classical Groups	331
9.5	Fourier Analysis for Non-central Measures	334
10	Comparison Techniques	335
10.1	The min-max Characterization of Eigenvalues	335
10.2	Comparing Dirichlet Forms Using Paths	336
10.3	Comparison for Non-symmetric Walks	339
	References	340

1 Introduction

This article surveys what is known about the convergence of random walks on finite groups, a subject to which Persi Diaconis gives a marvelous introduction in [27]. In the early twentieth century, Markov, Poincaré and Borel discussed the special instance of this problem associated with card shuffling where the underlying group is the symmetric group S_{52} . Two early references are to Emile Borel [15] and K.D. Kosambi and U.V.R. Rao [95]. The early literature focuses mostly on whether or not a given walk is ergodic: for card shuffling, ergodicity means that the deck gets mixed up after many shuffles.