

Contents

**I. General Ergodic Theory of Groups
of Measure Preserving Transformations**

1

II. Ergodic Theory of Smooth Dynamical Systems

103

**III. Dynamical Systems
on Homogeneous Spaces**

264

**IV. The Dynamics of Billiard Flows
in Rational Polygons**

360

**V. Dynamical Systems
of Statistical Mechanics and Kinetic Equations**

383

Subject Index

455

I. General Ergodic Theory of Groups of Measure Preserving Transformations

Contents

Chapter 1. Basic Notions of Ergodic Theory and Examples of Dynamical Systems (<i>I.P. Kornfeld, Ya.G. Sinai</i>)	2
§ 1. Dynamical Systems with Invariant Measures	2
§ 2. First Corollaries of the Existence of Invariant Measures. Ergodic Theorems	11
§ 3. Ergodicity. Decomposition into Ergodic Components. Various Mixing Conditions	18
§ 4. General Constructions	23
4.1. Direct Products of Dynamical Systems	23
4.2. Skew Products of Dynamical Systems	24
4.3. Factor-Systems	25
4.4. Integral and Induced Automorphisms	25
4.5. Special Flows and Special Representations of Flows	26
4.6. Natural Extensions of Endomorphisms	28
Chapter 2. Spectral Theory of Dynamical Systems (<i>I.P. Kornfeld, Ya.G. Sinai</i>)	30
§ 1. Groups of Unitary Operators and Semigroups of Isometric Operators Adjoint to Dynamical Systems	30
§ 2. The Structure of the Dynamical Systems with Pure Point and Quasidiscrete Spectra	33
§ 3. Examples of Spectral Analysis of Dynamical Systems	35
§ 4. Spectral Analysis of Gauss Dynamical Systems	36
Chapter 3. Entropy Theory of Dynamical Systems (<i>I.P. Kornfeld, Ya.G. Sinai</i>)	38
§ 1. Entropy and Conditional Entropy of a Partition	39
§ 2. Entropy of a Dynamical System	40
§ 3. The Structure of Dynamical Systems of Positive Entropy	43

§ 4. The Isomorphy Problem for Bernoulli Automorphisms and K -Systems	45
§ 5. Equivalence of Dynamical Systems in the Sense of Kakutani . . .	53
§ 6. Shifts in the Spaces of Sequences and Gibbs Measures	57
Chapter 4. Periodic Approximations and Their Applications. Ergodic Theorems, Spectral and Entropy Theory for the General Group Actions (<i>I.P. Kornfeld, A.M. Vershik</i>)	61
§ 1. Approximation Theory of Dynamical Systems by Periodic Ones. Flows on the Two-Dimensional Torus	61
§ 2. Flows on the Surfaces of Genus $p \geq 1$ and Interval Exchange Transformations	66
§ 3. General Group Actions	69
3.1. Introduction	69
3.2. General Definition of the Actions of Locally Compact Groups on Lebesgue Spaces	70
3.3. Ergodic Theorems	71
3.4. Spectral Theory	74
§ 4. Entropy Theory for the Actions of General Groups	76
Chapter 5. Trajectory Theory (<i>A.M. Vershik</i>)	80
§ 1. Statements of Main Results	80
§ 2. Sketch of the Proof. Tame Partitions	84
§ 3. Trajectory Theory for Amenable Groups	89
§ 4. Trajectory Theory for Non-Amenable Groups. Rigidity	91
§ 5. Concluding Remarks. Relationship Between Trajectory Theory and Operator Algebras	94
Bibliography	95
Additional Bibliography	101

II. Ergodic Theory of Smooth Dynamical Systems

Contents

Chapter 6. Stochasticity of Smooth Dynamical Systems.	
The Elements of KAM-Theory (<i>Ya.G. Sinai</i>)	106
§ 1. Integrable and Nonintegrable Smooth Dynamical Systems.	
The Hierarchy of Stochastic Properties of Deterministic Dynamics	106
§ 2. The Kolmogorov-Arnold-Moser Theory (KAM-Theory)	109
Chapter 7. General Theory of Smooth Hyperbolic Dynamical Systems (<i>Ya.B. Pesin</i>)	113
§ 1. Hyperbolicity of Individual Trajectories	113
1.1. Introductory Remarks	113
1.2. Uniform Hyperbolicity	114
1.3. Nonuniform Hyperbolicity	115
1.4. Local Manifolds	116
1.5. Global Manifolds	118
§ 2. Basic Classes of Smooth Hyperbolic Dynamical Systems.	
Definitions and Examples	118
2.1. Anosov Systems	118
2.2. Hyperbolic Sets	121
2.3. Locally Maximal Hyperbolic Sets	124
2.4. Axiom A-Diffeomorphisms	125
2.5. Hyperbolic Attractors. Repellers	126
2.6. Partially Hyperbolic Dynamical Systems	128
2.7. Mather Theory	129
2.8. Nonuniformly Hyperbolic Dynamical Systems.	
Lyapunov Exponents	131
§ 3. Ergodic Properties of Smooth Hyperbolic Dynamical Systems . .	133
3.1. u -Gibbs Measures	133
3.2. Symbolic Dynamics	135
3.3. Measures of Maximal Entropy	137
3.4. Construction of u -Gibbs Measures	137

3.5. Topological Pressure and Topological Entropy	138
3.6. Properties of u -Gibbs Measures	141
3.7. Small Stochastic Perturbations	142
3.8. Equilibrium States and Their Ergodic Properties	143
3.9. Ergodic Properties of Dynamical Systems with Nonzero Lyapunov Exponents	144
3.10. Ergodic Properties of Anosov Systems and of UPH-Systems	146
3.11. Continuous Time Dynamical Systems	149
§ 4. Hyperbolic Geodesic Flows	149
4.1. Manifolds with Negative Curvature	149
4.2. Riemannian Metrics Without Conjugate (or Focal) Points	153
4.3. Entropy of Geodesic Flows	156
4.4. Riemannian Metrics of Nonpositive Curvature	157
§ 5. Geodesic Flows on Manifolds with Constant Negative Curvature	158
§ 6. Dimension-like Characteristics of Invariant Sets for Dynamical Systems	161
6.1. Introductory Remarks	161
6.2. Hausdorff Dimension	161
6.3. Other Dimension Characteristics	164
6.4. Carathéodory Dimension Structure. Carathéodory Dimension Characteristics	167
6.5. Examples of C -structures and Carathéodory Dimension Characteristics	169
6.6. Multifractal Formalism	176
§ 7. Coupled Map Lattices	182
Additional References	190
Chapter 8. Billiards and Other Hyperbolic Systems (<i>L.A. Bunimovich</i>)	192
§ 1. Billiards	192
1.1. The General Definition of a Billiard	192
1.2. Billiards in Polygons and Polyhedrons	194
1.3. Billiards in Domains with Smooth Convex Boundary	196
1.4. Dispersing or Sinai Billiards	198
1.5. The Lorentz Gas and Hard Spheres Gas	206
1.6. Semi-dispersing Billiards and Boltzmann Hypotheses	206
1.7. Billiards in Domains with Boundary Possessing Focusing Components	209
1.8. Hyperbolic Dynamical Systems with Singularities (a General Approach)	215
1.9. Markov Approximations and Symbolic Dynamics for Hyperbolic Billiards	217
1.10. Statistical Properties of Dispersing Billiards and of the Lorentz Gas	219
1.11. Transport Coefficients for the Simplest Mechanical Models	222

§ 2. Strange Attractors	224
2.1. Definition of a Strange Attractor	224
2.2. The Lorenz Attractor	225
2.3. Some Other Examples of Hyperbolic Strange Attractors . . .	230
Additional References	231
Chapter 9. Ergodic Theory of One-Dimensional Mappings (<i>M.V. Jakobson</i>)	234
§ 1. Expanding Maps	234
1.1. Definitions, Examples, the Entropy Formula	234
1.2. Walters Theorem	237
§ 2. Absolutely Continuous Invariant Measures for Nonexpanding Maps	239
2.1. Some Examples	239
2.2. Intermittency of Stochastic and Stable Systems	241
2.3. Ergodic Properties of Absolutely Continuous Invariant Measures	243
§ 3. Feigenbaum Universality Law	245
3.1. The Phenomenon of Universality	245
3.2. Doubling Transformation	247
3.3. Neighborhood of the Fixed Point	249
3.4. Properties of Maps Belonging to the Stable Manifold of Φ	251
§ 4. Rational Endomorphisms of the Riemann Sphere	252
4.1. The Julia Set and Its Complement	252
4.2. The Stability Properties of Rational Endomorphisms	254
4.3. Ergodic and Dimensional Properties of Julia Sets	255
Bibliography	256

III. Dynamical Systems on Homogeneous Spaces

Contents

Chapter 10. Dynamical Systems on Homogeneous Spaces (<i>S.G. Dani</i>)	266
§ 1. Introduction	266
1.1. Measures on homogeneous spaces	266
1.2. Examples of lattices	268
1.3. Ergodicity and its consequences	271
1.4. Isomorphisms and factors of affine automorphisms	272
§ 2. Affine automorphisms of tori and nilmanifolds	273
2.1. Ergodic properties; the case of tori	273
2.2. Ergodic properties on nilmanifolds	275
2.3. Unipotent affine automorphisms	278
2.4. Quasi-unipotent affine automorphisms	280
2.5. Closed invariant sets of automorphisms	281
2.6. Dynamics of hyperbolic automorphisms	281
2.7. More on invariant sets of hyperbolic toral automorphisms	283
2.8. Distribution of orbits of hyperbolic automorphisms	285
2.9. Dynamics of ergodic toral automorphisms	286
2.10. Actions of groups of affine automorphisms	287
§ 3. Group-induced translation flows; special cases	289
3.1. Flows on solvmanifolds	289
3.2. Homogeneous spaces of semisimple groups	292
3.3. Flows on low-dimensional homogeneous spaces	295
§ 4. Ergodic properties of flows on general homogeneous spaces	297
4.1. Horospherical subgroups and Mautner phenomenon	298
4.2. Ergodicity of one-parameter flows	300
4.3. Invariant functions and ergodic decomposition	301
4.4. Actions of subgroups	303
4.5. Duality	304
4.6. Spectrum and mixing of group-induced flows	305

4.7. Mixing of higher orders	306
4.8. Entropy	307
4.9. K-mixing, Bernoullicity	308
§ 5. Group-induced flows with hyperbolic structure	309
5.1. Anosov automorphisms	309
5.2. Affine automorphisms with a hyperbolic fixed point	311
5.3. Anosov flows	312
§ 6. Invariant measures of group-induced flows	313
6.1. Invariant measures of Ad-unipotent flows	313
6.2. Invariant measures and epimorphic subgroups	316
6.3. Invariant measures of actions of diagonalisable groups	318
6.4. A weak recurrence property and infinite invariant measures	318
6.5. Distribution of orbits and polynomial trajectories	320
6.6. A uniform version of uniform distribution	321
6.7. Distribution of translates of closed orbits	323
§ 7. Orbit closures of group-induced flows	323
7.1. Homogeneity of orbit closures	323
7.2. Orbit closures of horospherical subgroups	325
7.3. Orbits of reductive subgroups	327
7.4. Orbit closures of one-parameter flows	328
7.5. Dense orbits and minimal sets of flows	330
7.6. Divergent trajectories of flows	332
7.7. Bounded orbits and escapable sets	333
§ 8. Duality and lattice-actions on vector spaces	335
8.1. Duality between orbits	335
8.2. Duality of invariant measures	336
§ 9. Applications to Diophantine approximation	338
9.1. Polynomials in one variable	338
9.2. Values of linear forms	338
9.3. Diophantine approximation with dependent quantities	339
9.4. Values of quadratic forms	340
9.5. Forms of higher degree	343
9.6. Integral points on algebraic varieties	343
§ 10. Classification and related questions	344
10.1. Metric isomorphisms and factors	345
10.2. Metric rigidity	346
10.3. Topological conjugacy	347
10.4. Topological equivalence	349
Bibliography	350

IV. The Dynamics of Billiard Flows in Rational Polygons

Contents

Chapter 11. The Dynamics of Billiard Flows in Rational Polygons of Dynamical Systems (<i>J. Smillie</i>)	360
§ 1. Two Examples	362
§ 2. Formal Properties of the Billiard Flow	364
§ 3. The Flow in a Fixed Direction	367
§ 4. Billiard Techniques: Minimality and Closed Orbits	369
§ 5. Billiard Techniques: Unique Ergodicity	372
§ 6. Dynamics on Moduli Spaces	374
§ 7. The Lattice Examples of Veech	377
Bibliography	380

V. Dynamical Systems of Statistical Mechanics and Kinetic Equations

Contents

Chapter 12. Dynamical Systems of Statistical Mechanics (<i>R.L. Dobrushin, Ya.G. Sinai, Yu.M. Sukhov</i>)	384
§ 1. Introduction	384
§ 2. Phase Space of Systems of Statistical Mechanics and Gibbs Measures	386
2.1. The Configuration Space	386
2.2. Poisson Measures	388
2.3. The Gibbs Configuration Probability Distribution	388
2.4. Potential of the Pair Interaction. Existence and Uniqueness of a Gibbs Configuration Probability Distribution	390
2.5. The Phase Space. The Gibbs Probability Distribution	393
2.6. Gibbs Measures with a General Potential	395
2.7. The Moment Measure and Moment Function	396
§ 3. Dynamics of a System of Interacting Particles	398
3.1. Statement of the Problem	398
3.2. Construction of the Dynamics and Time Evolution	400
3.3. Hierarchy of the Bogolyubov Equations	402
§ 4. Equilibrium Dynamics	403
4.1. Definition and Construction of Equilibrium Dynamics	403
4.2. The Gibbs Postulate	405
4.3. Degenerate Models	407
4.4. Asymptotic Properties of the Measures P_t	408
§ 5. Ideal Gas and Related Systems	408
5.1. The Poisson Superstructure	408
5.2. Asymptotic Behaviour of the Probability Distribution P_t as $t \rightarrow \infty$	410
5.3. The Dynamical System of One-Dimensional Hard Rods	411
§ 6. Kinetic Equations	412
6.1. Statement of the Problem	412
6.2. The Boltzmann Equation	415
6.3. The Vlasov Equation	419
6.4. The Landau Equation	420
6.5. Hydrodynamic Equations	421
Bibliography	423

Chapter 13. Existence and Uniqueness Theorems for the Boltzmann Equation (<i>N.B. Maslova</i>)	430
§ 1. Formulation of Boundary Problems. Properties of Integral Operators	430
1.1. The Boltzmann Equation	430
1.2. Formulation of Boundary Problems	434
1.3. Properties of the Collision Integral	435
§ 2. Linear Stationary Problems	437
2.1. Asymptotics	437
2.2. Internal Problems	438
2.3. External Problems	439
2.4. Kramers' Problem	441
§ 3. Nonlinear Stationary Problems	441
§ 4. Non-Stationary Problems	443
4.1. Relaxation in a Homogeneous Gas	443
4.2. The Cauchy Problem	444
4.3. Boundary Problems	445
§ 5. On a Connection of the Boltzmann Equation with Hydrodynamic Equations	446
5.1. Statement of the Problem	446
5.2. Local Solutions. Reduction to Euler Equations	448
5.3. A Global Theorem. Reduction to Navier-Stokes Equations	450
Bibliography	452