

## 2. Properties of Liquids and Gases

### 2.1 Properties of Liquids

Liquids are distinguished from solids by the fact that their particles are readily displaced. Whereas forces of finite magnitude are required to deform a solid, no force at all is required to alter the shape of a liquid, provided only that sufficient time is allowed for the change of shape to take place. When the shape is altered quickly, liquids do display a resistance, but this vanishes very quickly after the motion is finished. This ability of liquids to oppose a change in shape is called *viscosity*. We will discuss viscosity in depth in Chapter 4. As well as the usual liquids that are easy to move, there are also very viscous liquids whose resistance to change of shape is considerable, but which vanishes again at rest. Starting out from the viscous state, all phase transitions to (amorphous) solid bodies are possible. Heated glass, for example, passes through all possible transitions; in asphalt and similar substances these transitions occur at normal temperatures. For example, depending on the temperature, if a barrel of asphalt is tipped over, the asphalt will flow out within a few days or weeks. The mass that flows out forms a flat cake. Although it continually flows, one can walk on it without making footprints. Footprints will be left, however, if one stands still for a longer time on the asphalt. Hammering on the asphalt causes the mass to shatter like glass.

In the study of the *equilibrium* of liquids, we consider states of rest or sufficiently slow motion. The resistance to change of shape may then be set to zero, and we obtain *a definition of the liquid state: In a liquid in equilibrium, all resistance to change of shape is equal to zero.*

According to the kinetic theory of material, atoms or molecules are in constant motion. The kinetic energy of this motion is observed as heat. From this point of view, liquids differ from solids in that the particles do not oscillate about fixed positions, but rather more or less frequently swap places with neighboring particles. If the liquid is in a state of stress, such exchanges of place are favored. They cause the material to yield in the direction of the stress difference. In the state of rest this yielding causes the stress differences to vanish. During the change of shape, stresses arise that are larger the faster the change of shape takes place.

The gradual softening of amorphous bodies with increasing temperature may be explained as follows: If the body is heated, i.e., the energy of the

molecular motion is increased, initially some particles situated where the oscillation amplitudes just happen to be particularly large change place. On further heating, the exchange of place becomes more and more frequent, until eventually it occurs everywhere. For crystalline solid bodies the transition from a solid to a liquid state takes place discontinuously by melting, i.e., by the disintegration of the regular crystal structure.

A further property of liquids is their great resistance to change in volume. It is not possible to force 1 liter of water into a container half liter in size. If the same amount of water is placed in a container 2 liters in size, only half of the container is filled. However, water is not fully incompressible. At high pressures it can be pressed together by noticeable amounts (4% reduction in volume at a pressure of about 100 bar). Other liquids behave in a similar way.

## 2.2 State of Stress

We now consider more closely the state of stress of a liquid in equilibrium. We note that we can apply the general laws about the equilibrium of forces on a body to bodies of liquid too. In order to justify this, we define a particular *principle of solidification* based on the following idea: The equilibrium of an arbitrary movable system cannot be destroyed by subsequently fixing any moving parts. Therefore, we can imagine a certain part of a liquid in equilibrium to be solidified without destroying the equilibrium. The laws of equilibrium can be applied to the rigid part. Here we do not mean physical solidification, associated with change in volume and crystallization, etc., but rather ideal solidification without displacement or change of volume.

However, the detour via the rigid body is not really necessary. The laws of equilibrium in general mechanics are frequently derived by exploiting the idea of a *rigid body*. Yet these laws can still be applied to a *mass system at rest* with internal degrees of freedom of motion, which, however, are not used because of the equilibrium. As long as the system really is at rest, both approaches are equally valid. In the case of motion, the principle of solidification leads to difficulties, since nothing is solid. Because of the subsequent application in the dynamics of fluids, the essential ideas of this approach, used also in the science of the strength of materials, are briefly explained here.

We first note that forces are always interactions between masses. For example, if one mass  $m_1$  attracts another mass  $m_2$  with a force  $\mathbf{F}$ , this force  $\mathbf{F}$  also acts on  $m_1$  as the effect of  $m_2$ , as an attraction in the direction of  $m_2$ . The two forces act in opposite directions (Newton's principle of action and reaction). For a *system of masses* separated from other masses, we distinguish between two types of force. The *internal forces*, which act between two masses belonging to the system, and which therefore always act *opposite in pairs*, and the *external forces*, which act between each system mass and a mass situated outside the system, and which therefore occur only once in the system. If we

sum over all the forces acting on the masses in the system, the internal forces always cancel each other out in twos, so that only the external forces remain.

For the equilibrium of the system it is necessary that the sum of all the forces acting on each individual mass *vanish* (vector sum). If we sum this over all masses of the system, only the sum of all the *external forces* remains. Because each individual sum vanishes because of the equilibrium, the *sum* of the *external forces on the system* also *vanishes*. This law, which assumes no more about the mass system than that it is in equilibrium, is highly useful in many different applications. We obtain three statements:

$$\sum F_x = 0, \quad \sum F_y = 0, \quad \sum F_z = 0,$$

with the components  $F_x, F_y, F_z$  of the external forces in the  $x, y,$  and  $z$  directions.

As well as the above law, there is an analogous law for the torques of the external forces. Their sum also must vanish in equilibrium.

For both elastic solid bodies and liquid bodies we are interested in the *state of stress inside the body*. This arises via the internal forces that act between the smallest particles of the body. In general, we are content with knowing the average state in a region that already contains a large number of particles. Yet how should the internal forces be described if our laws give us statements only about external forces. *We must change them into external forces*. Imagine the body cut and one of the two pieces (labeled I in Figure 2.1) to be part of a mass system. Then all forces that came from a particle in region II and acted on a particle in region I, and which were previously internal forces, have now become external forces. If the whole body was in an external state of stress (indicated in Figure 2.1 by two arrows), internal stresses also occur. Imagining the cut carried out, forces act through the interface from the particles on the right of the cut on particles to the left of the cut. We add all these forces together to a resultant force, which then exactly maintains the equilibrium of the forces acting on part I. This gives us a clear statement on the resultant of the forces in the section. This approach could equally well have been applied to part II. We would have obtained an equally large resultant force pointing in the opposite direction (precisely the force acting from part I on part II).

By *stresses* we mean *forces per unit area in a section*. In the above example, we obtain the mean stress in the section when we divide the resultant force in the section due to equilibrium by the surface area of the section. We see that the *stress in a surface* is a vector, just as the force is.



**Fig. 2.1.** Forces on a mass system

The *cut principle*, i.e., the manner of transforming internal forces to external forces by imagining a cut, has further applications. With a number of planes of section through a body whose state of stress is to be investigated, we can select a small body (parallelepiped, prism, tetrahedron, etc.) and investigate its equilibrium. In the simplest case, all forces that hold a body in equilibrium are stress forces. From the equilibrium of such a body, we can derive several important laws; one is proved here as an example.

If we know the stress vectors for three planes of section that together form a corner of a body, then the stress vectors for all other planes of section are also known.

As proof, we cut the corner with a fourth plane, whose stress is to be determined. This gives rise to the tetrahedron shown in Figure 2.2. The forces 1, 2, and 3 are then obtained by multiplication of the given stress vectors with by surface areas of the associated triangles. There is only one direction and magnitude of the force 4, which maintains equilibrium with the sum of forces 1, 2, and 3. This force divided by the associated triangular surface area is the desired stress. For the calculation it is useful to select the surfaces 1, 2, and 3 as the coordinate planes (cf. Figure 2.2).

We point out that the *state of stress*, which represents the whole of the stress vectors in all possible cut directions through a point, can be related to an ellipsoid, and is therefore a tensor. According to the derived law, the state of stress in a point (and also its ellipsoid) is given if the stress vectors in three planes of section are known. Corresponding to the three principal axes of every ellipsoid, three orthogonal planes of section can be given for every state of stress to which the associated stress vectors are normal. The three stresses distinguished in this manner are called *principal stresses*.

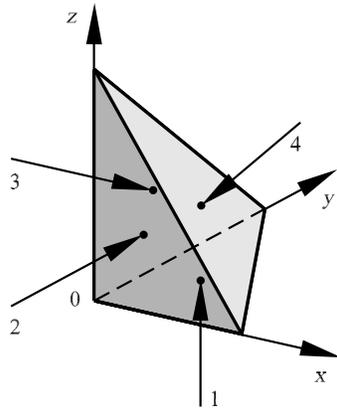
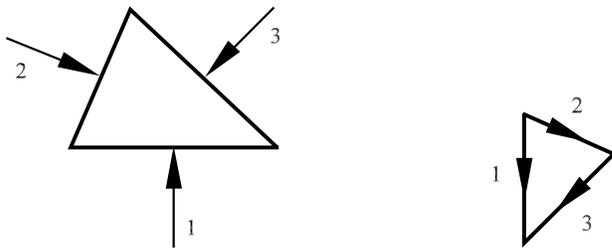


Fig. 2.2. Stress forces on a tetrahedron

## 2.3 Liquid Pressure

The state of stress of a liquid in equilibrium is particularly simple. A resistance to change of shape, thus against displacement of the particles against each other, can be compared to the friction of solid bodies. If there is no friction between two bodies that are in contact, the force must always be perpendicular to the contact surface between both bodies, so that no work is done by a sliding motion along this surface. Similarly, the absence of a resistance to change of shape is distinguished by the fact that the *stress*, here called the *pressure*, is *always perpendicular to a plane of section*. This property, that the pressure is perpendicular to the associated surface, can be taken as a *definition of the liquid state*. It is completely equivalent to the definition given in Section 2.1.

By a simple equilibrium approach, a further property of the liquid pressure may immediately be derived. We cut a small three-sided prism out of the liquid. The faces of the prism are perpendicular to the edges of the prism. Again we can imagine that the prism has solidified inside the liquid. We consider the equilibrium of the forces that act on the prism from the rest of the liquid. The pressure forces on the faces are equally large and directed opposite to each other. They therefore maintain the equilibrium and do not have to be considered further. The forces on the side surfaces are perpendicular to the surfaces, and are therefore in a plane perpendicular to the prism's edges. Figure 2.3 shows a front view of the prism with the forces, as well as the triangle that the forces must form so that they are in equilibrium. Since the sides of the force triangle are perpendicular to the sides of the prism, both triangles have the same angles and are therefore similar. This means that the three pressure forces behave like the associated prism sides. In order to determine the pressures per unit surface area, the pressure forces have to be divided by the respective prism surfaces. The prism surfaces all have the same height and are therefore in the same ratio to each other as their base lines and as the associated forces. Therefore, the *pressure per unit area* is equally large on all three prism surfaces. Since the prism was arbitrarily chosen, we can conclude that *the pressure at one point in the liquid is equally large in all directions*. The stress ellipsoid is a sphere in this case. In order to describe a



**Fig. 2.3.** Forces on the front side of a prism and force equilibrium

state of stress of this kind, also called the *hydrostatic state of stress*, we need only the numerical value of the pressure  $p$ . The pressure  $p$  means the force acting on a unit surface area.

### Pressure Distribution in a Liquid Without Gravitational Effects

Every liquid is heavy. In many cases, in particular at high pressures, the effect of gravity can be neglected, thereby simplifying matters greatly. Again we set up the force equilibrium on a prism, this time with a longitudinal shape. We consider the equilibrium change on displacement along the prism axis. The pressure varies with position. The cross-section of the prism is its front surface, here again assumed perpendicular to the axis of the prism, and is denoted by  $A$  (see Figure 2.4). This cross-section is assumed to be so small that the change in pressure within  $A$  can be neglected. If the pressure at one end of the prism is  $p_1$  and at the other  $p_2$ , the forces  $A \cdot p_1$  and  $A \cdot p_2$  act in opposite directions parallel to the axis of the prism. All pressure forces on the side faces of the prism are assumed to be perpendicular to these faces and are therefore also perpendicular to the prism axis. They do not contribute to the force component parallel to the prism axis, irrespective of how the pressure is distributed along it. Equilibrium demands that the forces  $A \cdot p_1$  and  $A \cdot p_2$  in the direction under consideration must balance each other. We must have

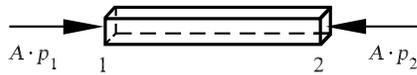
$$A \cdot p_1 = A \cdot p_2 \quad \text{or} \quad p_1 = p_2.$$

Since the position of the prism was chosen arbitrarily, *in the absence of gravity* (and other external forces) *the pressure at all positions in the liquid is equally large.*

If the liquid fills narrow, curved spaces, so that it is not possible to place a prism between two arbitrary points in the liquid, the above procedure can be repeated as often as necessary. We start out from point 1 to point 2, from this point in another direction to point 3, etc., until the required endpoint  $n$  is reached. From  $p_1 = p_2$ ,  $p_2 = p_3$ , etc., we then obtain  $p_1 = p_n$ .

Another, more elegant, approach is as follows: We imagine a large vessel into which the vessel under consideration fits and that is filled with liquid. After equilibrium has been reached, as much of the liquid as is necessary solidifies so that only the actual space of liquid remains. According to the principle of solidification in Section 2.2, there is no change in the state of forces. Therefore, everywhere in any narrow spaces in equilibrium, the same pressure is at hand.

In extremely narrow spaces, after a change in the liquid pressure, e.g., following an external stress, considerable time may pass until equilibrium is



**Fig. 2.4.** Pressure forces on a longitudinal prism

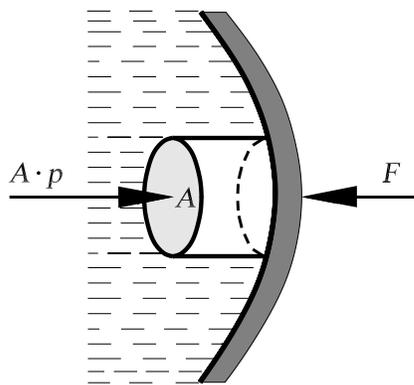
reached. For plastic potter's clay (consisting of very fine solid particles, with the spaces between filled with water), this time may be days, or, in the case of layers of clay in the earth, even years. During this time the water flows from positions of higher to those of lower pressure (see Section 4.2.8), while the solid frame yields elastically.

We summarize as follows: *The pressure in a liquid in equilibrium is everywhere perpendicular to the surface on which it acts and in the absence of gravity and other mass forces is everywhere and in all directions equally large.*

Whatever holds for the pressure inside the liquid is also true for the pressure on the walls of the vessel containing the liquid. To clarify this, we imagine a cut through the liquid very close to the wall and at some distance from it, and connect these two faces with a cylindrical surface perpendicular to the cuts (see Figure 2.5). The equilibrium of the body of water enclosed in this manner yields the force component  $\mathbf{F}$  that the section of wall perpendicular to the plane of section experiences, that is, the force  $A \cdot p$ . This approach has the advantage that we immediately see that uneven parts of the wall do not change the result. Figure 2.5 shows the force  $\mathbf{F}$  acting from the wall onto the body of liquid under consideration. The pressure force of the liquid on the wall has the opposite direction.

### Equilibrium of a Liquid

The effect of gravity on a given mass  $m$  is caused by a force of attraction to the center of the Earth of magnitude  $m \cdot g$ , where  $g$ , the acceleration of gravity, is equal to  $9.81 \text{ m/s}^2$  at our latitude. This value is not exact as the rotation of the Earth has been neglected. In fact, the force of gravity is due to the force of attraction and the centrifugal force. In the northern hemisphere, the direction of a plumb line intersects the axis of the Earth somewhat south of the center of the Earth.



**Fig. 2.5.** Pressure force on the wall of a vessel

The force  $m \cdot g$  is called the weight of the mass  $m$ . Because the amount of a liquid is frequently measured according to its volume, the *density*  $\rho$  is introduced for the *mass of a unit volume*. An amount of a liquid of volume  $V$  and density  $\rho$  therefore has a mass of  $\rho \cdot V$  and a *weight* of  $g \cdot \rho \cdot V$ . The product  $g \cdot \rho$  is therefore the weight of a volume unit and is called the specific weight  $\gamma$ . Because the strength of the gravitational acceleration  $g$  is not the same at all positions, the magnitude of the specific weight also varies from place to place. On the other hand, the density is independent of the strength of the gravitational force.

The *basic task of hydrostatics*, i.e., the study of the equilibrium of liquids, is to *determine the pressure distribution of a homogeneous liquid*.

We again consider the equilibrium of a bounded prism in a liquid to displacement in the axial direction and initially use the prism of Figure 2.4. Its axis is horizontal and is therefore at right angles to gravity. Therefore, the weight of the prism has no component in the axial direction, and so all the arguments from Section 2.3 may be repeated. Here again we obtain  $p_1 = p_2$ . By repeating this procedure for many prisms lined up with horizontal axes, we find that in all points in a horizontal plane the pressure must have the same value.

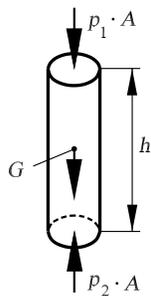
A relation between different horizontal planes is obtained by considering the equilibrium of a prism or cylinder with vertical axis to displacement in the vertical direction. In this case the weight of the prism has to be taken into account in the equilibrium of the forces. Corresponding to Figure 2.6, the pressure force  $p_1 \cdot A$  on the upper end face and the weight  $G = \gamma \cdot V = \gamma \cdot A \cdot h$  are directed downward. The pressure force  $p_2 \cdot A$  acts upward on the lower end face. Equilibrium requires that

$$\gamma \cdot A \cdot h + p_1 \cdot A = p_2 \cdot A.$$

Therefore,

$$p_2 - p_1 = \gamma \cdot h. \quad (2.1)$$

The pressure difference between the positions 1 and 2 is equal to the weight of the vertical column of liquid of cross-section 1 between them. Repeated



**Fig. 2.6.** Balance of forces on a vertical cylinder element

application of this procedure leads to the following result: *The pressure increases in the direction of the force of gravity by the amount  $\gamma$  for each unit of length. It is constant in every horizontal plane.*

If we introduce an  $x, y, z$  coordinate system whose  $z$  axis points vertically upward in the opposite direction to gravity, and if  $p_0$  is the pressure in the horizontal plane  $z = 0$ , the pressure  $p$  at an arbitrary position is given by

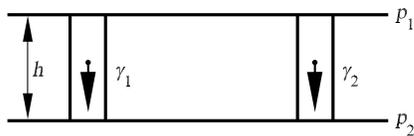
$$p = p_0 - \gamma \cdot z. \quad (2.2)$$

By applying the principle of solidification repeatedly, we see that this relation holds in large spaces filled with the liquid, in communicating vessels, in arbitrary pipe systems, in the gaps in gravel or sand, etc. The only assumption is a *homogeneous* connected liquid at rest.

The principle of solidification can also be used to determine the force that a body submerged in a liquid experiences due to liquid pressures. We first imagine the body replaced by liquid. The new section of liquid has the same shape as the body and has the same specific weight as the remaining liquid. It is kept in equilibrium by the pressure forces on its surface. The resultant of the pressure forces must point vertically upward, through the center of gravity of the new part of liquid. The size of this resultant force, called the lift, is equal to the product of the displaced volume  $V$  and the specific weight  $\gamma$  of the liquid. If we then imagine that the new part of the liquid solidifies, there is no change in the relations. Neither does anything change if another body of the same shape but a different weight is brought to the same position. This law was discovered by *Archimedes* and reads thus: *The loss of weight of a body submerged in a liquid is equal to the weight of the fluid it displaces.* If a body is weighed in a submerged state and in air, where it also experiences a small lift, there is a reduction in weight of  $G_{\text{liq}} - G_{\text{air}} = V \cdot (\gamma_{\text{liq}} - \gamma_{\text{air}})$ . This can be determined for a known specific weight  $\gamma_{\text{liq}}$  or a known volume  $V$ . The quantity  $\gamma_{\text{air}}$  can be computed using the concepts introduced in Section 2.5.

If the liquid is *inhomogeneous* (e.g., at different positions in a liquid with a nonuniform temperature distribution, salt solution with different salt content at different positions), the procedure with the prism with the horizontal axis can be applied without any change. Here, too, the pressure is the same in all horizontal planes. Two such horizontal planes a (not too large) distance  $h$  apart are selected (see Figure 2.7), with the upper plane at pressure  $p_1$  and the lower at pressure  $p_2$ . We consider two vertical prisms with height  $h$  and mean specific weights of  $\gamma_1$  and  $\gamma_2$  for the left and right prisms, respectively.

The balance of forces requires that on the left  $p_2 - p_1 = \gamma_1 \cdot h$  and on the right  $p_2 - p_1 = \gamma_2 \cdot h$ . This is possible only if  $\gamma_1 = \gamma_2$ . Otherwise, there



**Fig. 2.7.** Balance of forces on two horizontally displaced cylinder elements

would be no equilibrium, and the liquid would be set in motion. We can refine this approach by assuming the height  $h$  to be very small and carrying out the procedure for arbitrarily many pairs of neighboring horizontal planes. We obtain the result that *in an inhomogeneous liquid, equilibrium is possible only if the density is constant in every horizontal layer*. This result already contains the answer to the question of the equilibrium of two liquids of different densities that are layered above one another and do not mix. Their equilibrium requires that the interface must be a horizontal surface. We can directly apply the approach of Figure 2.7 to two homogeneous liquids layered above one another, whose interface is between the two horizontal planes and is initially unknown, and again we arrive at the same result.

Considering the *stability* of such a layering of liquids, we note that the liquid with the lower density always must be situated above the denser liquid. The reverse stratification is unstable. The smallest disturbance will put it into motion.

The proof of this can again be drawn from Figure 2.7. We assume a disturbed, slightly inclined interface between the two horizontal planes and determine the pressure differences in the interface. In the stable case, this inclination of the interface tends to decrease, whereas in the unstable case it tends to increase.

Similar statements hold for densities that vary continuously. The system is stable if the density everywhere decreases as we move upward. In contrast to the stable layered inhomogeneous liquid, the homogeneous liquid is a case of neutral equilibrium. Any parts of the liquid may be arbitrarily displaced without generating any forces that would disturb the equilibrium.

For the pressure distribution in the inhomogeneous liquid, for every layer in which the density is sufficiently inhomogeneous, equation (2.1) in differential form holds:

$$dp = -\gamma \cdot dz. \quad (2.3)$$

If  $\gamma$  is given as a function of the height  $z$ , integration leads to the relation

$$p = p_0 - \int_0^z \gamma \cdot dz. \quad (2.4)$$

## 2.4 Properties of Gases

Gases differ from liquids in that at large pressures they can be pressed together into a very small space. If more space become available than in the initial state, they always fill it uniformly, with a corresponding drop in pressure. Apart from this, their behavior is very similar to that of liquids. For gases at rest, all resistance to change of shape also vanishes, and they also have a certain viscosity to internal displacement. As long as there is no change

in volume, the behavior of a gas is qualitatively no different from that of a liquid that fills the same space without having a free surface.

The most important gas is the air in our atmosphere. Other gases have essentially the same behavior. As we will discuss in more detail in what follows, the air on the surface of the Earth is under approximately constant pressure of around 1 bar or  $10^5 \text{ N/m}^2$ . At higher altitudes the air pressure is lower (cf. Section 2.5).

There are several devices available to measure the air pressure (gas pressure). Devices that show pressure differences are called *manometers*. If they show absolute pressures of the surrounding gas, they are called *barometers*. Liquid columns can be used for both sorts of measurement (see Section 2.6). Devices for which the pressure to be measured acts on a spring are also frequently used. In order to measure the absolute pressure of the air, one can, for example, connect a metal can that has been pumped empty of air to a flexible lid with a strong spring, so that the tension of the spring just prevents the lid from being pushed in by the external air pressure. If this device is brought to a position with a different air pressure, the pressure change can be read from the deflection of the pointer (aneroid barometer, nowadays with digital display).

The law according to which the pressure of the gas changes for a given change in volume was first discovered by *R. Boyle* in 1662 and then independently by *Mariotte* in 1679. It is therefore called the *Boyle–Mariotte law*. According to this law, at constant temperature the pressure is inversely proportional to the volume. Therefore, if a fixed amount of gas is pressed together to half of its volume, its pressure doubles. If the volume is doubled, the pressure sinks by half. This law is expressed by the equation

$$p \cdot V = p_1 \cdot V_1, \quad (2.5)$$

where  $p_1$  is the initial pressure,  $V_1$  the initial volume, and  $p$  and  $V$  the values of these quantities for the gas in some given state.

The volume of a gas also changes greatly with the temperature. *Gay-Lussac* found in 1816 that the expansion of a gas for a change in temperature of  $1^\circ\text{C}$  at constant pressure is always  $1/273.2$  of its volume at  $0^\circ\text{C}$ . This is valid to good approximation for all gases and temperatures. This behavior is described by the equation

$$V = V_0 \cdot (1 + \alpha \cdot \vartheta), \quad (2.6)$$

where  $V_0$  is the volume at  $0^\circ\text{C}$ ,  $\vartheta$  the temperature in  $^\circ\text{C}$  and  $\alpha = 1/273.2^\circ\text{C}$  the *coefficient of expansion*. At moderate pressures, this value of  $\alpha$  is valid not only for air, but also to good approximation for other gases, like steam and helium.

Since equation (2.6) is independent of the pressure, it may be combined with equation (2.5). We therefore obtain an equation applicable at all pressures and temperatures:

$$p \cdot V = p_0 \cdot V_0 \cdot (1 + \alpha \cdot \vartheta). \quad (2.7)$$

Here  $p_0$  is an arbitrary but fixed initial pressure and  $V_0$  the volume at the initial pressure  $p_0$  and at 0 °C. Equation (2.7) is frequently called the *Mariotte–Gay-Lussac* law. It is also called the *equation of state*, since it connects the three state variables pressure, volume, and temperature. It is called the *equation of state of the ideal gas*, since the behavior of real gases deviates somewhat from this equation. For gases at normal densities these deviations may be neglected, but they are very important if the gas is greatly compressed, and particularly if the temperature is reduced so far that the gas begins to condense.

These deviations are treated in detail in thermodynamics. Here we mention only one of the deviations. According to equation (2.5), at very high pressures the gas volume is very small. Equation (2.7) can be used to calculate at which pressure the density of water, or that of gold, is reached. However, in reality this is impossible. There is a limiting volume below which the gas cannot be compressed, however large the pressure, i.e., a volume at which the molecules have attained their densest possible structuring. This fact can be included in equation (2.7), by writing

$$p \cdot (V - V') = p_0 \cdot (V_0 - V') \cdot (1 + \alpha \cdot \vartheta),$$

with the small limiting volume  $V'$ . For every finite  $p$ ,  $V$  is somewhat larger than  $V'$ . For volumes  $V$  that are large compared to  $V'$ , the results of this equation are essentially no different from those of equation (2.5) or (2.7).

As a gas is compressed, heat is generated. The *Boyle–Mariotte* law, which is valid only for constant temperatures, can be observed only if the gas has enough time during the compression to release the heat generated and to assume the surrounding temperature. The same is true for the cooling associated with expansion. If the gas is not given enough time to equalize its temperature differences, the ratio of the pressure to the initial pressure increases more strongly than the ratio of the volumes decreases. Thermodynamics states that in the case in which there is no exchange of the heat generated, i.e., when the compression or expansion takes place quickly, instead of equation (2.5) we have the equation

$$p \cdot V^\kappa = p_1 \cdot V_1^\kappa, \quad (2.8)$$

where  $\kappa = c_p/c_v$  is the ratio of the specific heat at constant pressure to the specific heat at constant volume. For dry air,  $\kappa = 1.4$ . Whereas a compression or expansion that obeys equation (2.5) is an *isothermal change of state*, a change according to equation (2.8) is called *adiabatic* compression or expansion. There is heating associated with adiabatic compression, and this can be calculated from equations (2.7) and (2.8), while cooling is associated with adiabatic expansion.

The behavior of gases discussed in this section can be explained by the assumption of gas kinetics that the molecules of the gas move at large velocities, colliding with each other and with the wall. The pressure is the summation of these collisions, and the temperature is the same as the kinetic energy of

the particles. The temperature increases on compression, as the velocity of the particles is increased due to elastic reflection as the walls move together.

## 2.5 Gas Pressure

The condition for the equilibrium of a gas is the same as that for the equilibrium of a liquid. The relations of the above section can therefore be carried over. In many cases, e.g., for moderate vertical extensions of a gas, the specific weight of the gas can be assumed to be spatially constant. Equations (2.1) and (2.2) of the previous section can be applied; i.e., the gas may be considered to be a homogeneous liquid. For greater vertical extensions (to the order of kilometers) this is no longer permissible. The pressure differences are so great here that because of the compressibility of the gas, the densities above and below are different. Temperature differences are also frequently important. Here the equation for inhomogeneous liquids must be used. Equation (2.3) is divided by  $\gamma$  and integrated. We obtain

$$\int_p^{p_0} \frac{dp}{\gamma} = z. \quad (2.9)$$

Depending on how the temperature depends on the height, this integral yields different results. The most important case is that of *constant temperature*. According to the *Boyle–Mariotte* law ( $p \cdot V = \text{const}$ ), the specific weight  $\gamma$  is directly proportional to the pressure:

$$\gamma = \gamma_0 \cdot \frac{p}{p_0}. \quad (2.10)$$

Therefore,

$$\int_p^{p_0} \frac{dp}{\gamma} = \frac{p_0}{\gamma_0} \cdot \int_p^{p_0} \frac{dp}{p} = \frac{p_0}{\gamma_0} \cdot \ln \left( \frac{p_0}{p} \right), \quad (2.11)$$

As can be seen from equation (2.1),  $p_0/\gamma_0$  is the height of a column of liquid with the constant specific weight  $\gamma_0$ , with pressure  $p_0$  at the lower end and a pressure of zero at the upper end. This height is called the *height of the uniform atmosphere*. With regard to the real atmosphere, it is nothing more than a computational quantity.

As an example we determine its numerical value. We therefore require the value of  $\gamma_0$ . In order to determine  $\gamma_0$  we proceed as follows: We weigh a container with a faucet out of which the air has been pumped. We then open the faucet and wait for the temperature to equalize, as the air in the container is initially heated by the work done by the external atmosphere as it flows into the container. We then weigh the container a second time. Since it was empty before and is now filled with air, its weight has increased

by the weight  $G$  of the air inside it. We then determine the volume  $V$  of the container, by, for example, pumping the air out of the container again, opening the faucet under water and again weighing the container filled with water. The measured quantities give us the value  $\gamma_0 = G/V$  associated with the pressure  $p_0$  on the ground. For every other ground pressure  $p_0$ ,  $\gamma_0$  can be calculated proportionally. Assuming that  $p_0$  is equal to 1 bar, for moderately damp air of temperature  $\vartheta$ , the Gay-Lussac law yields

$$\gamma = \frac{12.45}{1 + \alpha \cdot \vartheta} \text{ N/m}^3. \quad (2.12)$$

In dynamics, the density  $\rho = \gamma/g$  is used as a measure of the mass inertia. At room temperature, we can choose a mean value of  $11.8 \text{ N/m}^3$  for  $\gamma$ . With  $g = 9.81 \text{ m/s}^2$  we then obtain a mean value for  $\rho$  of  $1.20 \text{ N s}^2/\text{m}^4$ .

In order to compute  $p_0/\gamma_0$  in equation (2.11),  $p_0$  has to be expressed in the same mass system as  $\gamma_0$ . With  $1 \text{ bar} = 10^5 \text{ N/m}^2$ , we obtain

$$\frac{p_0}{\gamma_0} = \frac{100000}{12.45} \cdot (1 + \alpha \cdot \vartheta) = 8030 \cdot (1 + \alpha \cdot \vartheta).$$

The unit of  $p_0/\gamma_0$  is m. The height of the uniform atmosphere for moderately damp air is (independent of the pressure but dependent on the temperature)  $8030 \cdot (1 + \alpha \cdot \vartheta) \text{ m}$ . We set this equal to  $H_0$ . Equation (2.9) applied to two different heights yields

$$z_1 = H_0 \cdot \ln \left( \frac{p_0}{p_1} \right), \quad z_2 = H_0 \cdot \ln \left( \frac{p_0}{p_2} \right).$$

Therefore,

$$z_1 - z_2 = H_0 \cdot \ln \left( \frac{p_2}{p_1} \right). \quad (2.13)$$

This is the so-called *barometric height formula*. By inverting equation (2.13), we obtain the dependence of the pressure on the height:

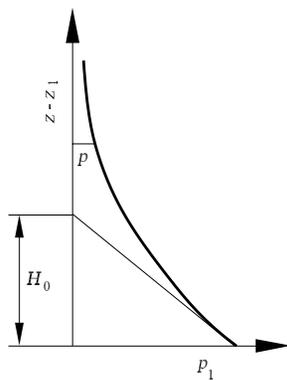
$$p = p_1 \cdot e^{-\frac{z-z_1}{H_0}}. \quad (2.14)$$

Considering the balance of the forces, in analogy to Figure 2.6, we see that the weight of a column of air with base area  $A$  that extends from position  $z$  upward to the edge of the atmosphere is equal to  $A \cdot p$ . Therefore,  $p$  is directly equal to the weight of the column of air with cross-section 1 situated above position  $z$ . Figure 2.8 shows equation (2.14) graphically. The pressure decreases continuously but ever more weakly with increasing height. For large heights it is equal to zero. The pressure decrease with height can be measured in the free atmosphere with a pressure-measuring device (barometer) on a tower or mountain. It can even be measured in a multistoried house. If the air temperatures are also measured, the observed pressure differences can be used to determine the difference in height. This method is used in aircraft to determine the altitude. If this height difference is known, this method can

also be used to determine the mean specific weight of the air layer situated between the two positions.

If the temperature of the mass of air is not constant, the height equation can still be applied to height sections in which the temperature differences are not very large. The height  $H_0$  associated with each height section is then determined for the mean value of the temperature in this section.

Finally, we turn to the question of when the equilibrium of a layered mass of gas is *stable* and when it is *unstable*. The condition that the specific weight of the upper layers must be smaller than that of the lower layers is not sufficient, because as a mass of gas moves upward or downward the pressure and thus the density of the mass of gas changes. The correct answer to the question is the following: The system is stable if a part of the gas at a greater height and at the new pressure is denser than its new surroundings, or if a part of the gas at a lower height and at the new pressure has a lower density than its new surroundings. In these cases the part of the gas will tend to return to its original position. There is a stratification (temperature distribution) in a mass of gas that corresponds to a homogeneous liquid, which therefore implies neutral equilibrium for the mass of gas. In order for this to hold, each part of the gas taken from an arbitrary position must have the same density as its surroundings after displacement, as if it had always belonged there. A part of a gas behaves adiabatically under a change of pressure as long as it has no possibility to exchange heat. If the stratification is such that pressure and density satisfy the equation of state (2.8) at all heights (i.e.,  $p$  is proportional to  $\gamma^\kappa$ ), every raised or lowered gas part always reaches a neighborhood with the temperature that it has itself due to its own adiabatic change of state. Therefore, it has no possibility to exchange heat with its surroundings. It can be shown that this *adiabatic stratification* has the following in common with a homogeneous liquid: It can be made homogeneous by strong mixing of an originally different type of stratification, such as an inhomogeneously layered salt solution.



**Fig. 2.8.** Pressure distribution in an atmosphere of constant temperature

In the air of the atmosphere, adiabatic stratification is characterized by the fact that the temperature decreases by 1 °C with an increase of height of 100 m. A lesser temperature decrease already indicates stability, while a temperature increase with height indicates even stronger stability. A larger temperature decrease than 1 °C per 100 m generally does not occur in the free atmosphere, since it would correspond to an unstable state. However, it is found close to the surface of the earth if the ground is hotter than the air. The air is then not in equilibrium, but rather is in motion with vertical upward and downward streams.

The pressure distribution in the adiabatically layered atmosphere can also be computed with equation (2.9), by setting  $\gamma = \gamma_0 \cdot (p/p_0)^{1/\kappa}$ . Integration yields

$$z = \frac{\kappa \cdot H_0}{\kappa - 1} \cdot \left( 1 - \left( \frac{p}{p_0} \right)^{\frac{\kappa-1}{\kappa}} \right) \quad \text{or} \quad p = p_0 \cdot \left( 1 - \frac{\kappa - 1}{\kappa} \cdot \frac{z}{H_0} \right)^{\frac{\kappa}{\kappa-1}}.$$

The equation of state  $p/\rho = R \cdot T$ , with the density  $\rho = \gamma/g$ , the absolute temperature  $T = (273.2 + \vartheta/1 \text{ °C}) \text{ K}$  and the gas constant  $R$ , with  $p_0/\gamma_0 = H_0$ , yields

$$\frac{R \cdot T}{g} = \frac{p}{\gamma} = H_0 - \frac{\kappa - 1}{\kappa} \cdot z, \quad \text{and so} \quad \frac{dz}{dT} = -H_0 \cdot \frac{\kappa}{\kappa - 1} \cdot \frac{R}{g}.$$

For moderately damp air,  $R/g = 29.4 \text{ m/K}$  and  $dz/dT = 102 \text{ m/K}$ .

If we replace  $\kappa$  in the above equations by a different number  $n$ , we obtain an interpolation formula that describes states of layering that actually occur in the atmosphere. These states of layering are called *polytropic*. For stable stratification,  $n < \kappa$ .

## 2.6 Interaction Between Gas Pressure and Liquid Pressure

As long as the pressure difference between the air in a container and the external air in the atmosphere is not too large, it can be measured with a U-tube manometer (cf. Figure 2.9). Neglecting the weight of the air, we obtain the following relations. At position A, the liquid pressure is equal to the air pressure  $p_1$  in the container. In the other limb of the U-tube, the pressure at the same height B is the same (communicating containers). Say the free liquid surface in this limb is at C. There the liquid pressure is equal to the pressure  $p_0$  of the atmosphere. According to the relations in Section 2.3,

$$p_1 = p_0 + \gamma \cdot h$$

if the height  $\overline{BC}$  is set equal to  $h$ . A U-tube filled with liquid is therefore suitable for measuring such pressure differences. It is used in various different forms. In order not to have to read the liquid heights at two positions (A and

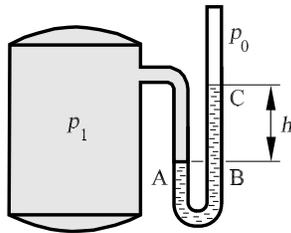
C in Figure 2.9), one of the limbs is frequently reshaped as a large pot in which the movement of the surface becomes very small (see Figure 2.10). To zero the device, both openings have to be connected to the atmosphere. For very small pressure differences the reading of the heights is refined, using for example a moveable microscope, or with a magnifying projection of a scale swimming on the surface of the liquid, according to *A. Betz*.

The use of the liquid manometer has led to a particular type of pressure units, where the pressure is expressed by the height of a liquid column. For example, 1 mm WC (water column, or WG water gauge) is equal to  $1 \text{ kp/m}^2 = 9.81 \text{ Pa}$ .

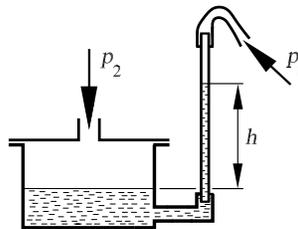
Water is not very suitable as a measurement liquid, since it wets the walls of the glass pipe very irregularly. All fat-soluble liquids (alcohol, toluol, xylol, etc.) are much more suitable. For larger pressure differences mercury is recommended, as in its pure state it permits very precise adjustment in a glass tube that is not too narrow. Because of the specific weight of  $133.370 \text{ N/m}^3$  at  $0^\circ \text{C}$ , 1 mm Hg (mercury) is equal to  $13.6 \text{ kp/m}^2 = 133.4 \text{ Pa}$ . The pressure unit 1 mm Hg is also called 1 torr, in honor of *Torricelli*. In recent times, membrane pressure gauges with digital data memory and piezopressure gauges that exploit the piezoelectric effect have been used.

If we pump some air out of the container in Figure 2.9, so that the pressure there becomes lower than the atmospheric pressure, the liquid in limb A of the U-tube will be higher than the liquid in limb B. Figure 2.11 shows a somewhat altered arrangement for the same experiment. The setup in Figure 2.9 is called an *overpressure manometer*, while that in Figure 2.11 is called a *vacuum manometer*. The pressure is measured from the height  $h$ .

Here we mention something about the history of pressure measurement: The question arose of how high a liquid can be sucked. In the middle ages, the rising of a column of liquid due to suction was explained by the idea of *horror vacui*, that “nature abhors a vacuum.” No investigations had been carried out into whether the *horror vacui* was arbitrarily strong, or whether it had a limit. It was the misfortune of Florentine pump makers, who built a water pump with the suction valve more than 10 m above the water surface and were unable to pump water as high as they wanted, that encouraged



**Fig. 2.9** Hydrostatic pressure measurement (U-tube manometer)



**Fig. 2.10** Liquid manometer

*Galileo* to look into the problem. Meanwhile, it was his pupil *Torricelli* who first recognized the facts, and this because of an experiment with mercury that he prompted his friend *Viviani* to perform in 1643. From our point of view, the answer to the question above is not difficult. Suction is merely compressing more weakly than the atmosphere compresses. The pressure in the container in Figure 2.11 is at its lowest when all the air has been pumped out of the container. Then it is equal to zero. The column of liquid can rise only so high that its height  $h$  corresponds to the air pressure  $p_0$  ( $h = p_0/\gamma$ ). *Viviani's* experiment was as follows: He took a glass tube two ells (120 cm) in length with a glass bubble blown on one end, and filled it completely with mercury from the other open end, closing this end with his finger. He then turned the tube upside down and placed the closed end in a flat container filled with mercury, and removed his finger. The column of mercury sank to a height of 1 1/4 ell (75 cm) above the surface of the mercury in the container and left an empty space behind. *Torricelli* correctly recognized that the mercury column retained the equilibrium with the outer air pressure. He observed that the mercury column did not always have the same height and concluded that the air pressure undergoes certain fluctuations. This fact is today of great importance for meteorology. *Torricelli* already concluded that the air pressure on a mountain must be higher than that in the valley, and that therefore the height of the mercury column is lower on the summit than down below. This was demonstrated several years later by *Perrier*, on the encouragement of *Pascal*, whereby he measured the height of the mercury column on the Puy de Dome and at the foot of this 975 m high mountain and noted a difference of 3 inches. The name *barometer* for this pressure gauge comes from *Pascal*. The word (derived from the Greek *barys*, meaning heavy) indicates that the weight of the air column above the liquid is what is measured.

At this point we mention another unit of pressure based on the barometer, the *physical atmosphere*. The mean level of a barometer at sea level is about

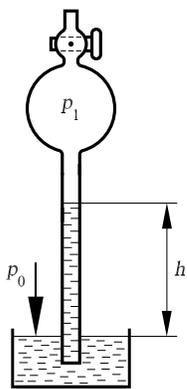


Fig. 2.11. Barometer

760 mm Hg. It has been agreed to define this barometer level at 0 °C mercury temperature as the normal state of the atmosphere and to call the associated air pressure one “physical atmosphere.” The qualifier “physical” is used because the technical atmosphere used by engineers is equal to 1 kp/cm<sup>2</sup>. Since the specific weight of mercury at 0 °C is equal to 13.595 p/cm<sup>3</sup> and 1 cm<sup>3</sup> therefore weighs 13.595 p, a mercury column of 76 cm therefore corresponds to a pressure of

$$76 \text{ cm} \cdot 13.595 \text{ p/cm}^3 = 1033.2 \text{ p/cm}^2 = 1.0132 \cdot 10^5 \text{ Pa.}$$

This pressure also corresponds to a water column of height 10.332 m (water barometer). The suction height of pumps must therefore be lower than this value.

Since the force of gravity plays a role in the definition of the physical atmosphere, and this does not have the same value at all positions on Earth, for greater precision in the definition of pressure units a particular value of the acceleration due to gravity  $g$  has been chosen. The value 980.665 cm/s<sup>2</sup> has been determined as the normal value of gravitational acceleration at the 45th degree of latitude at sea level. For a different acceleration due to gravity  $g$ , the pressure of the normal atmosphere is  $(1.0332 \cdot 980.665)/g$  local kiloponds per square centimeter. To get away from this somewhat arbitrary setting, a pressure unit was introduced to the CGS system: one million times the pressure unit 1 dyn/cm<sup>2</sup> is called the bar. At the normal value of gravitational acceleration, one bar corresponds to a mercury column of height 750.06 mm.

## 2.7 Equilibrium in Other Force Fields

In Sections 2.3 to 2.6, a homogeneous gravitational field was used; i.e., the acceleration due to gravity was assumed to be everywhere equally strong and orientated in the same direction. This assumption suffices for most applications. However, if we consider regions of Earth that are no longer small compared to Earth’s radius, the variations of the acceleration due to gravity in its magnitude and direction have to be taken into account. For a liquid at rest relative to a uniformly rotating container, in addition to the acceleration due to gravity, the centrifugal acceleration also has to be considered. In what follows we consider the quite general question of the equilibrium of a homogeneous or inhomogeneous liquid in a general force field, whose force per unit mass (i.e., acceleration) varies in strength and direction from place to place.

The considerations for a general force field lead directly from the ideas in Section 2.3. It follows from this section that the pressure cannot change in every direction perpendicular to the force field at hand (equilibrium of a small prism according to Figure 2.4 with the axis perpendicular to the direction of the force). Condensing all directions perpendicular to the force direction to one point, the pressure on the surface element perpendicular to

the force direction must be constant. For the case in which the adjoining surface elements can be integrated into one finite surface, i.e., when the force field has *normal surfaces*, the pressure is constant along all such normal surfaces. If a force field has no normal surface, then equilibrium is not possible in a liquid in this force field.

In contrast to the previous sections, where  $g$  denoted the strength of the gravitational field of the Earth,  $g$  will now denote the strength of a general force field. From the equilibrium at a small prism as in Figure 2.6 with height  $dh$  parallel to the force direction and pressure increase  $dp$ , we find that the pressure in the direction of the force increases according to the equation

$$dp = g \cdot \rho \cdot dh. \quad (2.15)$$

In the discussion below, we assume that the force field has normal surfaces. We consider two such normal surfaces with pressures  $p$  and  $p + dp$ . At two positions 1 and 2 in Figure 2.12, according to equation (2.15) we have on the one hand  $dp = g_1 \cdot \rho_1 \cdot dh_1$ , and on the other hand  $dp = g_2 \cdot \rho_2 \cdot dh_2$ . If  $\rho$  is either constant or a function of  $p$  (homogeneous liquid or homogeneous gas, cf. Sections 2.3 and 2.5), then  $p_1 = p_2$  and  $\rho_1 = \rho_2$ . This yields  $g_1 \cdot dh_1 = g_2 \cdot dh_2$ , where  $g \cdot dh$  is the work done by the force in the transition from one normal surface to the other. This work has the same value at all positions between the normal surfaces. The force field has a potential. The normal surfaces are therefore surfaces of constant potential. Introducing the potential  $U$  at a point with the equation

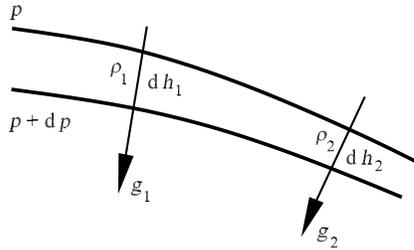
$$dU = -g \cdot dh \quad (2.16)$$

(the minus sign because in equation (2.15)  $dh$  in the direction of  $g$  is assumed positive), we obtain

$$dp = -\rho \cdot dU, \quad \text{or} \quad dU = -\frac{dp}{\rho}. \quad (2.17)$$

This yields the potential difference between two points A and B:

$$U_A - U_B = \int_A^B \frac{dp}{\rho}. \quad (2.18)$$



**Fig. 2.12.** Normal surfaces of a force field

In the case of a homogeneous liquid or a homogeneous gas assumed here, the right-hand side can be computed, and we obtain the pressure directly as a function of the potential. These results may be summarized as follows:

*In the case of a homogeneous liquid or a homogeneous gas, equilibrium is possible only if the force field has a potential. The surfaces of constant potential that lie perpendicular to the force are simultaneously surfaces of constant pressure. The pressure increases in the direction of the force. We have  $dp = -\rho \cdot dU$ .*

For an inhomogeneous liquid, it may happen that although  $g_1 \cdot dh_1$  is not equal  $g_2 \cdot dh_2$ , by suitable distribution of the density, we still have

$$\rho_1 \cdot g_1 \cdot dh_1 = \rho_2 \cdot g_2 \cdot dh_2.$$

It is seen that the equilibrium is unstable; as if the liquid were displaced along the normal surface, an action requiring no work, the distribution of the density is changed and the equilibrium disturbed. Therefore, if we want to restrict ourselves to stable states, we may consider only force fields that have a potential. However if  $g_1 \cdot dh_1$  is equal to  $g_2 \cdot dh_2$ , for equilibrium to exist we must have  $\rho_1 = \rho_2$ . Therefore, we can make the following assertion:

*A stable state of an inhomogeneous liquid is possible only if the force field has a potential. The surfaces of constant potential are simultaneously surfaces of constant pressure and constant density.*

Equations (2.17) and (2.18) may therefore be applied here too. The conditions for stability of the stratification are the same as those discussed for the homogeneous gravitational field in Sections 2.3 and 2.5.

Apart from magnetic force fields, the force fields that occur in physics almost always have a potential. However, the demand that the density  $\rho$  be constant on all surfaces of constant potential is of importance. This condition can be violated if the liquid or gas is locally heated, with a reduction in density at that region. In this case equilibrium is no longer possible, and the heated fluid and its surroundings are set into motion. This process comes to rest only if the warmer parts lie above the colder layers, and so the condition of constant density on surfaces of constant potential is again satisfied.

The free surface of a liquid or the interface between two immiscible liquids of different densities always follows a surface of constant potential. For this reason, surfaces of constant potential (*equipotential surfaces*) are also called *level surfaces* (free surface or level of an imaginary liquid). In surveying, the surface of the sea forms the level surface to which all heights are referred.

The discussions above will now be clarified in a simple example. Inside a container rotating uniformly about a vertical axis is a homogeneous liquid that is at rest relative to the rotating motion. We consider the equilibrium

of this liquid. We first determine an expression for the potential, which is additively made up of parts due to gravity and due to the centrifugal force.

Using cylindrical coordinates  $r$  and  $z$  (see Figure 2.13), we see that the contribution to the potential from gravity is  $U_1 = U_0 + g \cdot z$ , where  $g$  is the acceleration due to gravity and  $U_0$  an arbitrarily chosen starting potential. In order to determine the contribution to the potential from the centrifugal force, we note that the strength of the centrifugal force field is  $\omega^2 \cdot r$ , where  $\omega$  is the angular velocity with which the container and the liquid both rotate. Integrating in the direction of the centrifugal acceleration, i.e., in the direction of  $r$ , we obtain the second contribution to the potential:

$$U_2 = -\frac{\omega^2 \cdot r^2}{2}.$$

This yields the potential in a point of the liquid:

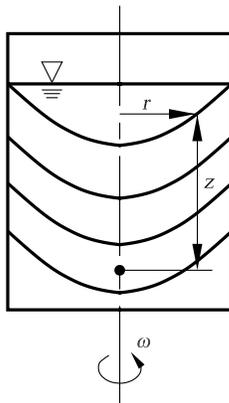
$$U = U_1 + U_2 = U_0 + g \cdot z - \frac{\omega^2 \cdot r^2}{2}.$$

The equipotential surfaces are found with the condition  $U = \text{const}$ :

$$z = \text{const} + \frac{\omega^2 \cdot r^2}{2 \cdot g}.$$

The free surfaces and all surfaces of equal pressure are paraboloids with the same parameter  $g/\omega^2$ . Integration of equation (2.17) leads to the relation  $p = p_0 - \rho \cdot U$  for the pressure. With  $\rho \cdot g = \gamma$  we obtain

$$p = \text{const} + \gamma \cdot \left( -z + \frac{\omega^2 \cdot r^2}{2 \cdot g} \right).$$



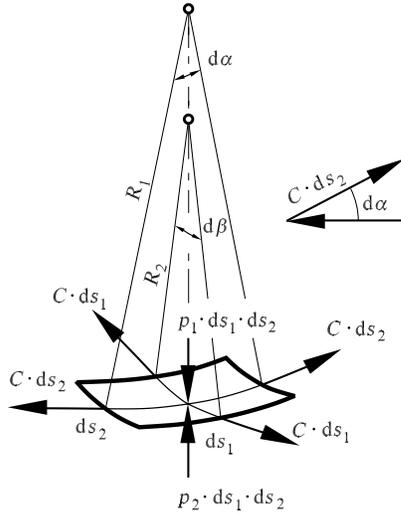
**Fig. 2.13.** Liquid in a rotating container

### 2.8 Surface Stress (Capillarity)

Free surfaces of liquids tend to shrink and form *minimal surfaces*. This behavior can be explained with a stress state in the surface taken on by a uniformly stretched thin skin. The origin of this tendency to shrink is as follows: Each liquid molecule close to the surface is pulled into the interior of the liquid by the attraction of the neighboring molecules (intermolecular forces). Because of this, only as many molecules as are absolutely necessary to form the surface remain on the surface. The same behavior is also found on interfaces between two liquids that do not mix. The stress that keeps the surface in equilibrium is called *surface stress*.

On flat interfaces the surface stress causes no pressure differences, since the resulting surface stress force is equal to zero. At curved surfaces pressure differences are necessary to establish equilibrium. We consider a small rectangle of a curved surface with sides of length  $ds_1$  and  $ds_2$  (see Figure 2.14). The pressure difference  $p_1 - p_2$  on the surface  $ds_1 \cdot ds_2$  leads to a force  $(p_1 - p_2) \cdot ds_1 \cdot ds_2$ . The surface stress is the force per unit length that keeps the surface in equilibrium. It has the magnitude  $C$  ( $C =$  capillary constant). Therefore, on the four edges of the rectangle we obtain two forces  $C \cdot ds_1$  on the sides  $ds_1$  and two forces  $C \cdot ds_2$  on the sides  $ds_2$ . The two forces on the sides  $ds_2$  are at an angle  $d\alpha = ds_1/R_1$  to each other. This leads to a resultant  $C \cdot ds_2 \cdot d\alpha = C \cdot ds_2 \cdot ds_1/R_1$ . The two other forces, which form the angle  $d\beta = ds_2/R_2$ , yield a resultant  $C \cdot ds_1 \cdot ds_2/R_2$ . From the equilibrium of the three forces we obtain

$$p_1 - p_2 = C \cdot \left( \frac{1}{R_1} + \frac{1}{R_2} \right). \tag{2.19}$$



**Fig. 2.14.** Surface stress and pressure on a curved liquid surface

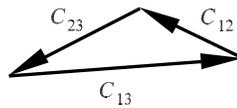
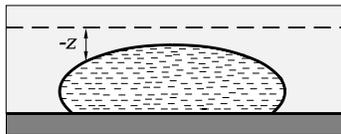
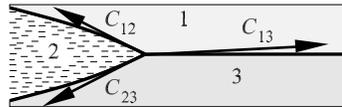
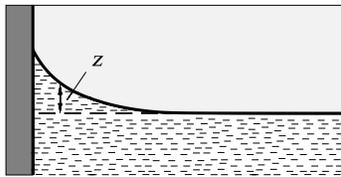
As seen in Figure 2.14,  $R_1$  and  $R_2$  are the radii of curvature of the curves of section of the surface with two orthogonal planes perpendicular to the tangential plane. Equation (2.19) leads to the geometric relation that the sum  $1/R_1 + 1/R_2$  is independent of the direction, since the pressure difference  $p_1 - p_2$  does not depend on the direction.

In *liquids* that are in equilibrium, the pressure dependent on the specific weight varies with height, according to the law  $p = p_0 - \gamma \cdot z$ . Therefore, at the interface of two liquids with specific weights  $\gamma_1$  and  $\gamma_2$ , we find that the associated pressures are  $p_1 = p_0 - \gamma_1 \cdot z$  and  $p_2 = p_0 - \gamma_2 \cdot z$ . With equation (2.19) we then obtain the relation between the curvature and the height at the interface:

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{\gamma_2 - \gamma_1}{C} \cdot z. \tag{2.20}$$

Figure 2.15 shows two examples of such surfaces. The capillary constant  $C$  can be determined by measurement of the geometries occurring.

It can be seen from equation (2.20) that for very small differences in the specific weights, we find an  $n$ -fold geometrically similar increase in the different surface forms ( $R_1$ ,  $R_2$  and  $z$  are  $n$  times as large) if the term  $(\gamma_2 - \gamma_1)/C$  is reduced by the factor  $1/n^2$ . For  $\gamma_2 = \gamma_1$  the effect of gravity vanishes. These surfaces are the so-called minimal surfaces. If for  $\gamma_2 - \gamma_1 \rightarrow 0$  we simultaneously set the plane  $z = 0$  at infinity, we find from equation (2.20) that  $1/R_1 + 1/R_2$  is constant. This result yields minimal surfaces with a given volume content, the simplest example of which is the sphere. These minimal surfaces may be obtained experimentally using soap films. In the interior of spherically shaped soap bubbles is an overpressure of magnitude  $p_1 - p_2 = 4 \cdot C/R$  (There are two surfaces of the soap solution in air to be taken into account, which is why the factor  $2 \cdot C$  instead of  $C$  is used in equation (2.19).)



**Fig. 2.15** Capillary surfaces of a liquid

**Fig. 2.16** Equilibrium of three surface stresses

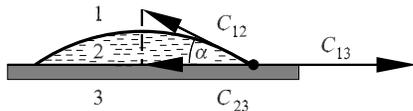
If three liquids meet along an edge, the balance of forces of the three surface stresses  $C_{12}$ ,  $C_{13}$ , and  $C_{23}$  yields certain angles at which the three interfaces join (see Figure 2.16). It may happen that  $C_{13}$  is larger than the sum of  $C_{12}$  and  $C_{23}$ . In this case no equilibrium is possible. For example, this happens when air, mineral oil, and water meet. The mineral oil then coats the entire surface, possibly with a very thin layer. This behavior is observed in the spreading of drops of motor oil on wet roads. If the oil is replaced by melted fat, this assumes the shape of flat lenses between the water and the air (globules of fat in soup). Figure 2.16 shows this case. If one of the three materials is solid, the balance of forces of the three surface stresses can be set up only with the components in the possible direction of displacement, parallel to the solid surface. Using the *wetting angle*  $\alpha$  (see Figure 2.17), we obtain  $C_{12} \cdot \cos(\alpha) + C_{23} = C_{13}$ , i.e.,

$$\cos(\alpha) = \frac{C_{13} - C_{23}}{C_{12}}. \tag{2.21}$$

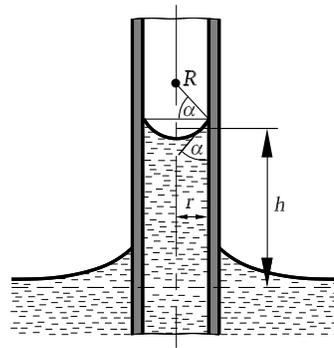
If  $C_{12}$  (surface stress at the interface of the two liquids 1 and 2) is already known and  $\alpha$  is measured, we can obtain the difference  $C_{13} - C_{23}$ . However,  $C_{13}$  and  $C_{23}$  cannot be individually determined. If the difference is negative, the angle  $\alpha$  is greater than  $\pi/2$  as with, for example, air, mercury, and glass. The lower picture in Figure 2.15 shows such a drop of mercury. The case  $C_{13} - C_{23} > C_{12}$  may also occur. Then the entire solid body is coated by liquid 2. This occurs in the case of petroleum.

Liquids are observed to rise considerably in narrow tubes. If  $r$  is the inner radius of the tube, then, simplifying the liquid surface as a spherical shell ( $r$  small compared to  $h$ ), we see from Figure 2.18 that the spherical radius is  $R = r/\cos(\alpha)$ , with the wetting angle  $\alpha$ . Therefore, according to equation (2.20), we obtain

$$h = \frac{2 \cdot C_{12}}{\gamma_2 - \gamma_1} \cdot \frac{\cos(\alpha)}{r}. \tag{2.22}$$



**Fig. 2.17** Wetting angle on a solid surface



**Fig. 2.18** Capillary rise in a tube

The height  $h$  can become very large if  $r$  is very small (suction effect of blotting paper, fine clay, etc.).

In equation (2.22) we can eliminate  $\cos(\alpha)$  using equation (2.21) and multiply both sides by  $\pi \cdot r^2 \cdot (\gamma_2 - \gamma_1)$ . This yields the equation

$$(\gamma_2 - \gamma_1) \cdot \pi \cdot r^2 \cdot h = (C_{13} - C_{23}) \cdot 2 \cdot \pi \cdot r.$$

The weight of the column of liquid, reduced by its lift, is equal to the resulting tensile force on the tube wall. If the *tensile force* is negative, i.e.,  $\alpha > \pi/2$  as in the case of mercury,  $h$  becomes negative (Figure 2.18 reflected in the horizontal plane). For wetted surfaces  $C_{13} - C_{23}$  may be replaced by  $C_{12}$ . Then  $\cos(\alpha) = 1$ ; i.e.,  $\alpha = 0$ . This yields the maximum value of  $h$ . On measurement of  $h$  and  $r$  we obtain

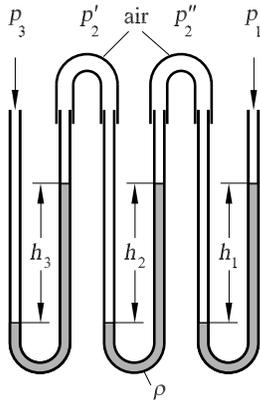
$$C_{12} = \frac{1}{2} \cdot (\gamma_2 - \gamma_1) \cdot h \cdot r.$$

Another method of determining  $C_{12}$  is the measurement of capillary waves, to be discussed in Section 4.1.8.

Values of $C_{12}$ at 20 °C:	water to air	0.073 N/m,
	oil to air	0.025 to 0.030 N/m,
	mercury to air	0.472 N/m.

## 2.9 Problems

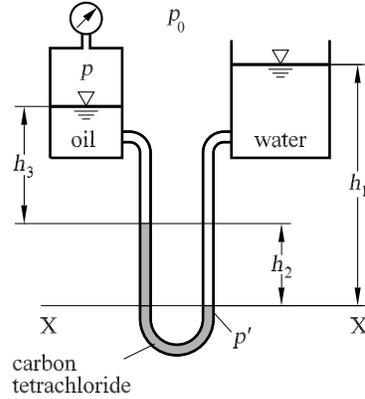
### 2.1



Three identical U-tubes are connected in a row. In each U-tube is a liquid with density  $\rho$ . The levels of the liquids show the height differences  $h_1$ ,  $h_2$ , and  $h_3$ . The effect of gravity on the air may be neglected. How great is the pressure difference  $\Delta p = p_3 - p_1$  between the free ends of the first and third tubes?

$$\Delta p = p_3 - p_1 = \rho \cdot g \cdot (h_1 + h_2 + h_3).$$

2.2



An open water container and a container that is closed to the atmosphere by a manometer are connected by a U-tube, whose lower part is filled with carbon tetrachloride (tet) ( $\text{CCl}_4$ ). The height of the water column (density of water  $\rho_w = 1000 \text{ kg/m}^3$ ) is  $h_1 = 0.4 \text{ m}$ , the column of oil (density of oil  $\rho_{\text{oil}} = 950 \text{ kg/m}^3$ ) has the height  $h_3 = 0.13 \text{ m}$ , and the height  $h_2$  of the  $\text{CCl}_4$  column is  $h_2 = 0.1 \text{ m}$ .

What is the density  $\rho_{\text{tet}}$  of the  $\text{CCl}_4$  filling if an excess pressure compared to the atmospheric pressure of  $1200 \text{ N/m}^2$  is read from the manometer?

$$\rho_{\text{tet}} = 1541.76 \text{ kg/m}^3.$$

2.3

The pressure  $p_0$  and the temperature  $T_0$  are known for the atmosphere at sea level  $z = 0$  (specific gas constant of air  $R = 287 \text{ m}^2/(\text{s}^2 \cdot \text{K})$ ,  $p_0 = 101300 \text{ N/m}^2$ ,  $T_0 = 283 \text{ K}$ ).

(a) Assuming that the state of the gas in the atmosphere changes isothermally, determine the dependence of the pressure and the density of the atmosphere on the height  $z$ .

$$p = p_0 \cdot e^{-\frac{z}{H_0}}, \quad \rho = \rho_0 \cdot e^{-\frac{z}{H_0}}, \quad H_0 = \frac{R \cdot T_0}{g} \quad .$$

(b) Assuming that the state of the gas in the atmosphere changes polytropically, determine the dependence of the pressure and the density of the atmosphere on the height  $z$ :

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^n, \quad \frac{p}{p_0} = \left(1 - \frac{n-1}{n} \cdot \frac{z}{H_0}\right)^{\frac{n}{n-1}}, \quad \frac{\rho}{\rho_0} = \left(1 - \frac{n-1}{n} \cdot \frac{z}{H_0}\right)^{\frac{1}{n-1}} \quad .$$

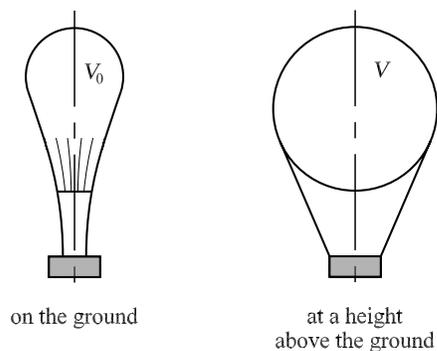
2.4

A balloon is suspended in an isothermal atmosphere (air pressure on the ground  $p_0 = 1.013 \text{ bar}$ , air density on the ground  $\rho_0 = 1.225 \text{ kg/m}^3$ ) at a height  $z_0 = 500 \text{ m}$ . How far will the balloon sink if a change in the weather

causes the air density on the ground to change to  $\rho'_0 = 1.0 \text{ kg/m}^3$  while the air pressure remains the same? The volume  $V$  of the balloon is not to change as the height varies.

$$z_x = H'_0 \cdot \left[ \ln \left( \frac{\rho'_0}{\rho_0} \right) + \frac{z_0}{H_0} \right], \quad \Delta z = 272.41 \text{ m.}$$

## 2.5



A stratospheric balloon is partially filled with the buoyant gas hydrogen  $\text{H}_2$  on the ground. As the balloon rises, it inflates with an increase in volume of the filling. This leads to an additional lift. On the ground, the balloon has volume  $V_0 = 450 \text{ m}^3$ , while its maximum volume is  $V_1 = 1400 \text{ m}^3$ .

(a) What is the greatest possible weight of the load  $G_{\text{max}}$  to be lifted (the balloon itself is part of the weight, but the buoyant gas is not) if the stratospheric balloon is to reach a maximum height of  $z_{\text{max}} = 1.2 \text{ km}$  in a polytropic atmosphere? On the ground, the air pressure is  $p_0 = 1.013 \text{ bar}$  and the air density is  $\rho_0 = 1.234 \text{ kg/m}^3$ . The density of hydrogen  $\rho_{\text{H}_2,0}$  in the balloon has the value  $\rho_{\text{H}_2,0} = 0.087 \text{ kg/m}^3$  on the ground. The temperature  $T_{1\text{km}} = 280 \text{ K}$  at an altitude of 1 km, and the specific gas constant of the air  $R = 287 \text{ m}^2/(\text{s}^2 \cdot \text{K})$  are also known.

$$G_{\text{max}} = 3955.8 \text{ N.}$$

(b) At what height  $z_1$  does the balloon reach its largest volume  $V_1 = 1400 \text{ m}^3$ ? Until the maximum volume is reached, the hydrogen in the balloon is to have the same temperature and pressure as the atmosphere at all heights.

$$z_1 = H_0 \cdot \frac{n}{n-1} \cdot \left[ 1 - \left( \frac{V_0}{V_1} \right)^{n-1} \right], \quad z_1 = 10224.1 \text{ m.}$$

## 2.6

A number of small solids are moving on the surface of a liquid. Show that the surface stress causes the solids to move toward each other, whether they are wetted by the liquid or not. They move away from each other if one solid is wetted and the other is not wetted by the liquid.

**2.7**

How much work  $W$  must be done to atomize a volume  $V$  of liquid into spherically shaped droplets of radius  $R$ ? The surface energy of the volume  $V$  before the atomization is assumed to be negligible.

$$W = \frac{3C}{R} \cdot V.$$