Contents

Dedication **v**

Acknowledgements **vii**

1 Overview **1**
1.1 Introduction 1
1.2 Historical Overview 2
1.3 Plan of the Book 8
1.4 References 12

2 Fundamental Concepts **13**
2.1 Introduction 13
2.2 Conventional Perturbation Theory 15
2.3 Integrability 18
2.3.1 Noether's Theorem 19
2.3.2 Hidden Symmetries 21
2.3.3 Poincaré Surface of Section 24
2.4 Nonlinear Resonance and Chaos 28
2.4.1 Single-Resonance Hamiltonians 28
2.4.2 Two-Resonance Hamiltonian 32
2.5 KAM Theory 35
2.6 The Definition of Chaos 38
2.6.1 Lyapounov Exponent 39
2.6.2 KS Metric Entropy and K-Flows 43

2.7 Time-Dependent Hamiltonians 47
2.8 Conclusions 54
2.9 Problems 55
2.10 References 56

3 Area-Preserving Maps 58
3.1 Introduction 58
3.2 Twist Maps 60
3.2.1 Derivation of a Twist Map from a Torus 60
3.2.2 Generating Functions 62
3.2.3 Birkhoff Fixed Point Theorem 63
3.2.4 The Tangent Map 64
3.2.5 Homoclinic and Heteroclinic Points 66
3.3 Melnikov Distance 71
3.4 Whisker Maps 74
3.5 The Standard Map 77
3.5.1 Rational and Irrational Orbits 78
3.5.2 Accelerator Modes 81
3.6 Scaling Behavior of Noble KAM Tori 85
3.6.1 Rational Approximates 85
3.6.2 Scaling Properties of Twist Maps 88
3.7 Renormalization in Twist Maps 95
3.7.1 Integrable Twist Map 95
3.7.2 Nonintegrable Twist Map 98
3.7.3 The Universal Map 100
3.8 Bifurcation of M-Cycles 102
3.8.1 Some General Properties 102
3.8.2 The Quadratic Map 103
3.8.3 Scaling in the Quadratic de Vogelaere Map 105
3.9 Cantori 112
3.10 Diffusion in Two-Dimensional Twist Maps 117
3.10.1 Effect of Cantori 117
3.10.2 Diffusion in the Standard Map 123
3.11 Conclusions 126
3.12 Problems 130
3.13 References 130

4 Global Properties 134
4.1 Introduction 134
4.2 Important Models 136
4.2.1 Delta-Kicked Rotor (Standard Map) 136
4.2.2 The Duffing Oscillator 139
4.2.3 Driven Particle in Infinite Square-Well Potential . 140
4.2.4 Driven One-Dimensional Hydrogen 141
4.3 Renormalization Map 146

4.3.1 The Paradigm Hamiltonian 147
4.3.2 The Renormalization Map 149
4.3.3 Fixed Points of the Renormalization Map 153
4.4 Application of Renormalization Predictions 158
4.4.1 Driven Square-Well System 158
4.4.2 Duffing Oscillator 161
4.5 Scattering Chaos 162
4.6 Stochastic Tiling 167
4.6.1 Delta-Kicked Harmonic Oscillator 167
4.6.2 Two Primary Resonances Model 168
4.7 Arnol'd Diffusion 170
4.7.1 Resonance Networks 171
4.7.2 Numerical Observations 174
4.7.3 Diffusion Along Separatrix Layers 175
4.7.4 Diffusion Coefficient 180
4.7.5 Some Applications 182
4.8 Conclusions 186
4.9 Problems 186
4.10 References 187

5 Random Matrix Theory 189
5.1 Introduction 189
5.2 Ensembles 192
5.2.1 Gaussian Ensembles 193
5.2.2 Circular Ensembles 196
5.3 Cluster Functions 197
5.3.1 Cluster Expansion of the Probability Densities . . 197
5.3.2 Probability Densities and Quaternion Determinants 199
5.3.3 Cluster Functions for Gaussian Ensembles 200
5.3.4 Cluster Functions for Circular Ensembles 202
5.4 Eigenvalue Number Density 204
5.4.1 Eigenvalue Number Density for Gaussian Ensembles 204
5.4.2 Eigenvalue Number Density for Circular Ensembles 210
5.5 Eigenvalue Correlations – Δ_3-Statistic 211
5.5.1 Δ_3-Statistic – General Expressions 211
5.5.2 Δ_3-Statistics for Gaussian Ensembles 214
5.5.3 Δ_3-Statistics for Circular Ensembles 220
5.6 Eigenvalue Nearest Neighbor Spacing Distribution (GOE) 223
5.6.1 Eigenvalue Spacing Distributions (N=2) 223
5.6.2 Nearest Neighbor Spacing Distribution ($N \to \infty$) . 225
5.6.3 Approximate Nearest Neighbor Spacing Distributions for GOE ($N \to \infty$) 227
5.7 Eigenvector Statistics – Gaussian Ensembles 228
5.7.1 General Properties 229
5.7.2 Distribution of Eigenvector Components (GOE) . 231

5.7.3 Distribution of Eigenvector Components (GUE) . 232
5.7.4 Gaussian Symplectic Ensemble 233
5.8 Conclusions 234
5.9 Problems 235
5.10 References 235

6 Bounded Quantum Systems 237
6.1 Introduction 237
6.2 Quantum Integrability 240
6.3 Symmetries and Degeneracy 242
6.4 Quantum Billiards 246
6.4.1 The Rectangular Billiard 247
6.4.2 The Stadium 249
6.4.3 The Sinai Billiard 257
6.4.4 The Ripple Billiard 258
6.5 The Quantized Baker's Map 264
6.6 Time Average as an Invariant 269
6.7 Integrable and Nonintegrable Spin Systems 271
6.7.1 Classical Spin Models 271
6.7.2 Quantum Spin Models 278
6.7.3 Two-Dimensional Clusters with N Spin $\frac{1}{2}$ Objects 281
6.8 Anharmonic Oscillators 282
6.8.1 Polynomial Anharmonicity 282
6.8.2 Coupled Morse Oscillators 284
6.9 Conclusions 288
6.10 Problems 289
6.11 References 289

7 Manifestations of Chaos in Quantum Scattering Processes 293
7.1 Introduction 293
7.2 Scattering Theory 297
7.2.1 Hamiltonian 298
7.2.2 Energy Eigenstates 301
7.2.3 The Reaction Matrix 302
7.2.4 The Scattering Matrix 305
7.3 Wigner-Smith and Partial Delay Times 310
7.3.1 Delay Time of a Wave Packet 310
7.3.2 Delay Times for Multichannel Scattering 311
7.3.3 Delay Times and Complex Poles 314
7.4 Scattering Theory and GOE 316
7.4.1 The Average S-Matrix 317
7.4.2 When Does a GOE Hamiltonian Yield a COE S-Matrix? 318
7.4.3 S-Matrix Correlation Function (GOE) 323

7.4.4 Delay Time Density 326
7.5 S-Matrix Correlation Functions (COE) 327
7.6 Green's Function and S-Matrix 330
7.6.1 The Green's Function 330
7.6.2 Green's Function for Quantum Waveguide 331
7.6.3 Transmission Amplitudes and the Green's Function 332
7.7 Absorption Spectrum and Green's Function 336
7.8 Experimental Observation of RMT Predictions 339
7.8.1 Experimental Nuclear Spectral Statistics 339
7.8.2 Experimental Molecular Spectral Statistics 341
7.9 Conclusions 342
7.10 Problems 344
7.11 References 345

8 Semiclassical Theory—Path Integrals 348
8.1 Introduction 348
8.2 Green's Function and Density of States 350
8.3 The Path Integral 351
8.3.1 The General Case, $\hat{H} = \hat{T} + \hat{V}$ 351
8.4 Semiclassical Approximation 354
8.4.1 Method of Stationary Phase 354
8.4.2 The Semiclassical Green's Function 355
8.4.3 Conjugate Points 360
8.5 Energy Green's Function 362
8.5.1 General Expression 362
8.5.2 Density of States 368
8.6 Δ_3-Statistic for a Rectangular Billiard 372
8.6.1 Energy Green's Function for a Rectangular Billiard 372
8.6.2 Density of States for the Rectangular Billiard . . 374
8.6.3 Semiclassical Expression for the Δ_3-Statistic . . . 377
8.7 Gutzwiller Trace Formula 379
8.7.1 Response Function for Chaotic Systems 380
8.8 Anisotropic Kepler System 385
8.9 Diamagnetic Hydrogen 389
8.9.1 The Model 390
8.9.2 Absorption Cross Section 392
8.9.3 Experiment 393
8.9.4 Semiclassical Cross Section 395
8.10 Conclusions 398
8.11 Problems 399
8.12 References 399

9 Time-Periodic Systems 401
9.1 Introduction 401
9.2 Floquet Theory 403

9.2.1 Floquet Matrix 403
9.2.2 Floquet Hamiltonian 405
9.3 Nonlinear Quantum Resonances 406
9.3.1 Two Primary Resonance Model 406
9.3.2 Floquet Eigenstates 408
9.3.3 Quantum Resonance Overlap 409
9.3.4 Floquet Eigenvalue Nearest Neighbor Spacing . . 413
9.4 Eigenvalue Avoided Crossings and Wave Function Delocalization 414
9.4.1 Classical Driven Square-Well System 414
9.4.2 Quantum Driven Square-Well System 416
9.4.3 Avoided Crossings and Delocalization 418
9.4.4 High Harmonic Radiation 420
9.5 Dynamical Tunneling in Atom Optics Experiments . . . 422
9.5.1 Hamiltonian for Atomic Center-of-Mass 423
9.5.2 Average Momentum of Cesium Atoms 425
9.5.3 Floquet Analysis of Tunneling Oscillations 425
9.6 Quantum Renormalization 428
9.6.1 Paradigm Schrödinger Equation 430
9.6.2 Higher-Order Resonances 432
9.6.3 Quantum Renormalization Map 434
9.6.4 Stable Manifold 436
9.6.5 Scaling Functions 438
9.6.6 Scaling of Localization Lengths 439
9.7 Quantum Delta-Kicked Rotor 442
9.7.1 The Schrödinger Equation for the Delta-Kicked Rotor 442
9.7.2 KAM-like Behavior of the Quantum Delta-Kicked Rotor 443
9.7.3 The Floquet Map 444
9.7.4 Spectral Statistics 446
9.8 Dynamic Anderson Localization: Delta-Kicked Rotor . . 447
9.8.1 Tight-Binding Model for the Delta-Kicked Rotor 448
9.8.2 Diffusion Coefficient and Localization Length . . 452
9.8.3 Atom Optics Realization of the Delta-Kicked Rotor 453
9.9 Microwave-Driven Hydrogen 457
9.9.1 Experimental Apparatus 457
9.9.2 One-Dimensional Approximation 460
9.10 Dynamic Anderson Localization - Microwave-Driven Hydrogen 465
9.10.1 Diffusion in Microwave-Driven Hydrogen 465
9.10.2 Experimental Observation of Dynamic Localization 467
9.11 Conclusions 469
9.12 Problems 471
9.13 References 471

10 Stochastic Manifestations of Chaos **474**
10.1 Introduction . . . 474
10.2 Brownian Motion in Two Space Dimensions . . . 474
10.3 Random Walk in Two Space Dimensions . . . 478
10.4 Time-Periodically Driven Brownian Motion in One Space Dimension . . . 481
10.4.1 Schrödinger-like Equation . . . 482
10.5 Conclusion . . . 485
10.6 References . . . 485

A Classical Mechanics **486**
A.1 Newton's Equations . . . 486
A.2 Lagrange's Equations . . . 487
A.3 Hamilton's Equations . . . 487
A.4 The Poisson Bracket . . . 488
A.5 Phase Space Volume Conservation . . . 489
A.6 Action-Angle Coordinates . . . 489
A.7 Hamilton's Principal Function . . . 491
A.8 References . . . 492

B Simple Models **493**
B.1 The Pendulum . . . 493
B.1.1 Libration—Trapped Orbits ($E_0 < g$) . . . 494
B.1.2 Rotation—Untrapped Orbits ($E_0 > g$) . . . 495
B.2 Double-Well Potential . . . 496
B.2.1 Trapped Motion—($E_o < 0$) . . . 497
B.2.2 Untrapped Motion —($E_o > 0$) . . . 498
B.3 Infinite Square-Well Potential . . . 499
B.4 One-Dimensional Hydrogen . . . 501
B.4.1 Zero Stark Field . . . 501
B.4.2 Nonzero Stark Field . . . 503

C Renormalization Integral **505**
C.1 $\nu = N =$ Integer . . . 505
C.2 $\nu = \frac{N}{M} \neq$Integer . . . 507
C.3 References . . . 509

D Moyal Bracket **510**
D.1 The Wigner Function . . . 510
D.2 Ordering of Operators . . . 512
D.3 Moyal Bracket . . . 513
D.4 References . . . 514

E Symmetries and the Hamiltonian Matrix **515**
E.1 Space-Time Symmetries . . . 516

E.1.1 Continuous Symmetries 516
E.1.2 Discrete Symmetries 518
E.2 Structure of the Hamiltonian Matrix 520
E.2.1 Space-Time Homogeneity and Isotropy 520
E.2.2 Time Reversal Invariance 521
E.3 References 524

F Invariant Measures **525**
F.1 General Definition of Invariant Measure 525
F.1.1 Invariant Metric (Length) 526
F.1.2 Invariant Measure (Volume) 526
F.2 Hermitian Matrices 527
F.2.1 Real Symmetric Matrix 527
F.2.2 Complex Hermitian Matrices 530
F.2.3 Quaternion Real Matrices 532
F.2.4 General Formula for Invariant Measure of Hermitian Matrices 534
F.3 Unitary Matrices 534
F.3.1 Symmetric Unitary Matrices 536
F.3.2 General Unitary Matrices 537
F.3.3 Symplectic Unitary Matrices 538
F.3.4 General Formula for Invariant Measure of Unitary Matrices 539
F.3.5 Orthogonal Matrices 539
F.4 References 540

G Quaternions **541**
G.1 References 545

H Gaussian Ensembles **546**
H.1 Vandermonde Determinant 547
H.2 Gaussian Unitary Ensemble (GUE) 549
H.3 Gaussian Orthogonal Ensemble (GOE) 550
H.4 Gaussian Symplectic Ensemble (GSE) 558
H.5 References 563

I Circular Ensembles **564**
I.1 Vandermonde Determinant 565
I.2 Circular Unitary Ensemble (CUE) 566
I.3 Circular Orthogonal Ensemble (COE) 567
I.4 Circular Symplectic Ensemble (COE) 572
I.5 References 576

J Volume of Invariant Measure for Unitary Matrices **577**
J.1 References 582

K Lorentzian Ensembles **583**
K.1 Normalization of ΛOE 583
K.2 Relation Between COE and ΛOE 584
K.3 Equivalence of COE and ΛOE When $N \to \infty$. 585
K.4 Invariance of ΛOE under Inversion 586
K.4.1 Robustness of ΛOE under Integration 587
K.5 References 587

L Grassmann Variables and Supermatrices **588**
L.1 Grassmann Variables 588
L.2 Supermatrices 590
L.2.1 Transpose of a Supermatrix 590
L.2.2 Hermitian Adjoint of a Supermatrix 591
L.2.3 Supertrace of a Supermatrix 591
L.2.4 Determinant of a Supermatrix 592
L.3 References 594

M Average Response Function (GOE) **595**
M.1 Gaussian Integral for $(\mathrm{Det}[e\bar{1}_N - \bar{H}_N])^{-1}$ 596
M.2 Gaussian Integral for $\mathrm{Det}[e\bar{1}_N - \bar{H}_N]$ 597
M.3 Gaussian Integral for Response Function Generating Function 598
M.4 Expectation Value of the Generating Function (Part 1) . 600
M.5 The Hubbard-Stratonovitch Transformation 601
M.6 Expectation Value of the Generating Function (Part 2) . 604
M.7 Average Response Function Density 608
M.7.1 Saddle Points for the Integration over a 610
M.7.2 Saddle Points for the Integration over w 612
M.7.3 Asymptotic Expression ($N \to \infty$) for the Average Response Function Density 614
M.7.4 Wigner Semicircle Law 616
M.8 References 617

N Average S-Matrix (GOE) **618**
N.1 S-Matrix Generating Function 618
N.2 Average S-Matrix Generating Function 619
N.3 Saddle Point Approximation 622
N.3.1 Case I: $\frac{E}{2} < \mu$ 623
N.3.2 Case II: $\frac{E}{2} > \mu$ 624
N.4 Integration over Grassmann Variables 624
N.5 References 627

O Maxwell's Equations for 2-d Billiards **628**
O.1 References 631

P Lloyd's Model **632**
P.1 Localization Length 632
P.2 References 637

Q Hydrogen in a Constant Electric Field **638**
Q.1 The Schrödinger Equation 638
Q.1.1 Equation for Relative Motion 639
Q.1.2 Solution for $\lambda_0 = 0$ 641
Q.2 One-Dimensional Hydrogen 642
Q.3 References 644

Subject Index **645**

Author Index **667**