CHAPTER 2: STANDARD PRICING RESULTS UNDER DETERMINISTIC AND STOCHASTIC INTEREST RATES

Along with providing the way uncertainty is formalized in the considered economy, we establish in this chapter the assumptions that will be adopted throughout Part I of this book and the general principles governing asset pricing (§1), then the relationship between the spot and the forward prices of a risky asset (§2), and lastly that between the spot and the futures prices (§3). Any dividend (or coupon, or convenience yield) will always be assumed both continuous and deterministic.

2.1. GENERAL SETTING AND MAIN ASSUMPTIONS

In this framework, individuals can trade continuously on a frictionless and arbitrage free financial market until time τ_E, the horizon of the economy. A locally riskless asset and a number n of pure default-free discount bonds sufficient to complete the market are traded. The latter pay one dollar each at their respective maturities, respectively T_j, $j = 1,..., n$, with $T_{j-1} < T_j < \tau_E$. The following sets of assumptions provide the necessary details.

A*ssumption 1: Dynamics of the primitive assets.*

- At each date t, the price $P(t,T_j)$ of a discount bond whose maturity is T_j , $j = 1,..., n$, is given by :

$$P\left(t, T_j\right)=\exp\left[-\int_t^{T_j} f\left(t, s\right)ds\right] \tag{1}$$

where f(t,s) is the instantaneous forward rate (thereafter the forward rate) prevailing at time t for date s, with $t < s \leq \tau_E$.

- The instantaneous spot rate (thereafter the spot rate) is r(t) = f(t,t). Agents are allowed to trade on a money market account yielding this continuous bounded spot rate. Let B(t) denote its value at time t, with B(0)=1. Then:

$$B(t) = \exp\left[\int_0^t r(u)du\right] \tag{2}$$

- To give our model some additional structure, we assume, following Heath, Jarrow and Morton (1992), that the forward rate is the solution to the following stochastic differential equation:

$$df(t,s) = \mu(t,s)dt + \Sigma(t,s)'dZ(t) \tag{3}$$

where $\mu(.)$, the drift term, and $\Sigma(t,s)$, the K-dimensional vector of diffusion parameters (volatilities), are assumed to satisfy the usual conditions[1] such that (3) has a unique solution, and "$'$" denotes a transpose. $Z(t)$ is a K-dimensional Brownian motion defined on the complete filtered space $(\Omega, F, \{F_t\}_{t \in [0,\tau_E]}, P)$, where Ω is the state space, F is the σ-algebra representing measurable events, and P is the actual (historical) probability. The forward rate is adapted to the augmented filtration generated by this Brownian motion. This filtration is denoted by $F \equiv \{F_t\}_{t \in [0,\tau_E]}$ and is assumed to satisfy the usual conditions [2]. The initial value of the forward rate, f(0,s), is observable and given by the initial yield curve prevailing on the market.

- Since the latter is arbitrage free, the drift term $\mu(t, s)$ in equation (3) is a specific function of the forward rates volatility $\Sigma(t, s)$ that involves the market prices of risk associated with the K sources of uncertainty. This is the so-called « drift condition ». If markets are complete, Proposition 3, on p. 86, of HJM (1992) establishes that this relationship between the drift and the volatility is unique. More precisely, it states that there exists a unique vector of *market prices of risk* $\phi(t)$ such that:

$$\mu(t,s) = -\sum_{j=1}^{K} \sigma_j(t,s)\left[\phi_j(t) - \int_t^s \sigma_j(t,u)du\right]$$

for all $s \in [0, \tau_E]$ and $t \in [0, s]$, where $\sigma_j(t,s)$ is the j[th] element of $\Sigma(t,s)$ and $\phi_j(t)$ is the j[th] element of $\phi(t)$.

Assumption 2: Absence of frictions and of arbitrage opportunities.

- The assumption of absence of arbitrage opportunities in a frictionless financial market leads to the First Theorem of asset pricing theory. Since Harrison and Kreps (1979), this assumption is in effect known to be tantamount to assuming the existence of a probability measure, defined with respect to a given numeraire and equivalent to P, such that the prices of all risky assets, deflated by this numeraire, are martingales[3].

- Now, applying Itô's lemma to (1) given (3) yields the stochastic differential equation satisfied by the discount bonds:

$$\frac{dP(t,T_j)}{P(t,T_j)} = \mu_P(t,T_j)dt + \Sigma_P(t,T_j)'dZ(t) \qquad j=1,...,n. \tag{4}$$

where $\Sigma_P(t,T_j)$ is the K-dimensional vector of the volatilities associated with the relative price changes of the discount bond maturing at T_j, $P(t,T_j)$. This vector is functionally related to the vector $\Sigma(.)$ of the forward rates volatilities. The drift $\mu_P(t,T_j)$ plays no particular role here, but could easily be computed as the sum of the riskless rate plus a risk premium that depends on the bond maturity date T_j. Note that, since the market is complete, we have $n \geq K$.

- The absence of arbitrage implies the existence of a martingale measure Q, associated with the locally riskless asset (more precisely the money market account B(t)) as the numeraire, and such that its Radon-Nikodym derivative with respect to the historical probability is equal to:

$$\left.\frac{dQ}{dP}\right|_{F_t} \equiv \eta(t) = \exp\left\{-\int_0^t \phi(s)'dZ(s) - \frac{1}{2}\int_0^t \phi(s)'\phi(s)ds\right\}$$

where $\phi(s)$ is the vector of market prices of risk. The latter is a predictable, F_t-adapted, process satisfying Novikov's condition $E^P\left[\exp\left(\frac{1}{2}\int_0^T \|\phi(s)\|^2 ds\right)\right] < \infty$, where the expectation E^P is taken under the true measure P. The probability Q is generally, but somewhat misleadingly, called the "*risk-neutral*" measure.

Now, the Second Theorem of asset pricing theory relates the uniqueness of the martingale measure Q to the completeness of the financial market. A market is complete if any risky asset can be replicated by a portfolio of existing (traded) assets, so that its price is unique. This requires that there exist, in addition to the riskless asset, as many risky assets traded on the market as there are fundamental sources of risk in the economy (the dimension of the Brownian motion, for instance, as here). A portfolio of existing assets, whose composition changes (in general) continuously over time, can then replicate any contingent claim. Therefore, the latter can be fairly priced. It also means that any risk can be perfectly hedged by the appropriate combination of existing assets.

When markets are complete, there exists only one martingale measure associated with a given numeraire. Thus, in our setting, Q is unique. When markets are incomplete, however, such a result breaks down and for each numeraire there exists more than one martingale measure. Consequently, there is possibly an infinite number of prices for each contingent claim. We will indicate in the sequel which results require completeness and which do not.

- Under Q, the price of a pure discount bond follows the dynamic process:

$$\frac{dP(t,T_j)}{P(t,T_j)} = r(t)dt + \Sigma_P(t,T_j)'\,d\hat{Z}(t) \qquad j = 1,\dots,n. \tag{5}$$

where $\hat{Z}(t)$ is a K-dimensional Brownian motion under Q. Using Girsanov's theorem, it is related to the Brownian motion Z by:

$$d\hat{Z}(t) = dZ(t) + \phi(t)dt$$

Integrating (5) then yields:

$$P(t,T_j) = P(0,T_j)\exp\left[\int_0^t\left(r(s) - \frac{1}{2}\Sigma_P(s,T_j)'\,\Sigma_P(s,T_j)\right)ds + \int_0^t \Sigma_P(s,T_j)'\,d\hat{Z}(s)\right]$$

$$j = 1,\dots,n. \tag{6}$$

- Consider any traded asset with payoff S(T) at time T and no intermediate cash flow. Its price today is S(t). T is assumed to be smaller than T_1 so that all the discount bonds are "long-lived" assets. By construction, we have:

$$\frac{S(t)}{B(t)} = E^Q\left[\frac{S(T)}{B(T)}\middle|F_t\right] \tag{7}$$

where $E^Q[.|F_t]$ denotes the conditional expectation under Q based upon all information available at time t.

- While the locally riskless money market account B(t) is by far the most extensively used numeraire, another one is frequently used when interest rates are stochastic. Recall that the sole objective of choosing a particular numeraire is to ease the mathematical burden of computing conditional expectations. Instead of choosing Q and its associated numeraire B(t), it is convenient to adopt the measure Q^T and its associated numeraire P(t,T), the value of the zero-coupon bond of maturity T. This measure, first used formally by Jamshidian (1987, 1989), is known as the "*T-forward-neutral*" probability. Under Q^T, the price of any non-dividend-paying asset deflated by P(t,T) is a martingale:

$$\frac{S(t)}{P(t,T)} = E^{Q^T}\left[\frac{S(T)}{P(T,T)}\middle| F_t\right] \tag{8}$$

which can be rewritten:

$$\frac{S(t)}{P(t,T)} = E^{Q^T}\left[S(T)\middle| F_t\right] \tag{9}$$

Formally, Q^T is defined as:

$$\left.\frac{dQ^T}{dQ}\right|_{F_t} = \frac{P(t,T)B(0)}{P(0,T)B(t)} = \frac{P(t,T)}{P(0,T)B(t)} \equiv \varepsilon_t^T, \forall t \le T \tag{10}$$

where $B(t) = \exp\left[\int_0^t r(s)ds\right]$ is a stochastic process, as is P(t,T). Obviously, if interest rates are deterministic, the Radon-Nikodym derivative dQ^T/dQ is always equal to one in absence of arbitrage opportunities and the "T-forward-neutral" and "risk-neutral" measures are identical.

Assumption 3: Admissible strategies.

Our final set of assumptions concerns the economic agents' behavior. Each individual who can freely trade on the continuously open financial market adopts a portfolio strategy that consists in choosing the appropriate number of units of the locally riskless asset (the value of the money market account) and of each and every discount bond. Such strategies are assumed to be admissible[4], and in particular self-financing.

2.2. FORWARD PRICES

- Let G(t,T) be the price of the maturity-T forward contract written on one particular asset, say S(t). For the moment, this asset does not pay out dividends. When negotiated at t, the value of the forward contract is zero, so that:

$$0 = E^{Q}\left[\frac{G(t,T)-S(T)}{B(T)}\middle|F_t\right] \tag{11}$$

Rearranging terms yields:

$$G(t,T)E^{Q}\left[\frac{1}{B(T)}\middle|F_t\right] = E^{Q}\left[\frac{S(T)}{B(T)}\middle|F_t\right] \tag{12}$$

From the definition of Q, we have:

$$E^{Q}\left[\frac{1}{B(T)}\middle|F_t\right] = E^{Q}\left[\frac{P(T,T)}{B(T)}\middle|F_t\right] = \frac{P(t,T)}{B(t)} \text{ and } E^{Q}\left[\frac{S(T)}{B(T)}\middle|F_t\right] = \frac{S(t)}{B(t)}$$

Substituting into (12) yields the *cash-and-carry relationship without dividends:*

$$G(t,T) = \frac{S(t)}{P(t,T)} \tag{13}$$

This result can be derived alternatively, and even more rapidly, using the zero-coupon bond price P(t,T) as the numeraire. Under the Q^T-forward neutral probability, we have:

$$0 = E^{Q^T}\left[\frac{G(t,T)-S(T)}{P(T,T)}\middle|F_t\right] \tag{14}$$

and thus:

$$G(t,T) = E^{Q^T}\left[\frac{S(T)}{P(T,T)}\middle|F_t\right] = \frac{S(t)}{P(t,T)} \tag{15}$$

Remark that it is easy to go from (11) to (14), given the relationship between the two measures. Indeed, Bayes formula states that:

$$E^{Q^T}\left[X(T)\middle|F_t\right] = \frac{E^{Q}\left[X(T)Y(T)\middle|F_t\right]}{E^{Q}\left[Y(T)\middle|F_t\right]} \text{ where } \left.\frac{dQ^T}{dQ}\right|_{F_t} = Y(t)$$

It follows that:

$$E^{Q^T}\left[\frac{G(t,T)-S(T)}{P(T,T)}\middle|F_t\right] = \frac{E^{Q}\left[\frac{G(t,T)-S(T)}{P(T,T)}\frac{P(T,T)}{B(T)P(0,T)}\middle|F_t\right]}{E^{Q}\left[\frac{P(T,T)}{B(T)P(0,T)}\middle|F_t\right]}$$

Simplifying some terms yields:

$$E^{Q^T}\left[\frac{G(t,T)-S(T)}{P(T,T)}\middle|F_t\right]=\frac{E^{Q}\left[\frac{G(t,T)-S(T)}{B(T)}\middle|F_t\right]}{E^{Q}\left[\frac{P(T,T)}{B(T)}\middle|F_t\right]}$$

Since

$$E^{Q}\left[\frac{P(T,T)}{B(T)}\middle|F_t\right]=\frac{P(t,T)}{B(t)}$$

we have:

$$E^{Q^T}\left[\frac{G(t,T)-S(T)}{P(T,T)}\middle|F_t\right]=\frac{B(t)}{P(t,T)}E^{Q}\left[\frac{G(t,T)-S(T)}{B(T)}\middle|F_t\right]$$

Given that (14) holds, (11) must hold too.

We stress here that the cash-and-carry formula (13) or (15) is valid irrespective of whether the market is complete or not. Absence of frictions and arbitrage opportunities is enough. What is only required for (13) to hold is that agents are allowed to trade the asset underlying the forward contract and a discount bond maturing at time T.

- The preceding findings must be amended when the underlying asset pays out dividends. Assume a continuous dividend rate process $d_s(t)$. S(t) is the ex-dividend asset price. This price being the current value of the claim to the asset at date T plus all dividends up to T, it follows that, under the risk-neutral measure Q, we must have:

$$\frac{S(t)}{B(t)}=E^{Q}\left[\frac{S(T)e^{\int_t^T d_s(s)ds}}{B(T)}\middle|F_t\right] \tag{16}$$

Now, it is still true that:

$$
\begin{aligned}
0 &= E^Q\left[\frac{G(t,T)-S(T)}{B(T)}\middle|F_t\right] = E^Q\left[\frac{G(t,T)}{B(T)}\middle|F_t\right] - E^Q\left[\frac{S(T)}{B(T)}\middle|F_t\right] \\
&= G(t,T)E^Q\left[\frac{1}{B(T)}\middle|F_t\right] - E^Q\left[\frac{S(T)}{B(T)}\middle|F_t\right] \\
&= G(t,T)E^Q\left[\frac{P(T,T)}{B(T)}\middle|F_t\right] - E^Q\left[\frac{S(T)}{B(T)}\middle|F_t\right] \\
&= G(t,T)\frac{P(t,T)}{B(t)} - E^Q\left[\frac{S(T)}{B(T)}\middle|F_t\right]
\end{aligned}
$$

Therefore:

$$G(t,T) = \frac{B(t)}{P(t,T)}E^Q\left[\frac{S(T)}{B(T)}\middle|F_t\right]$$

The last term is no longer a martingale under Q in general, unless the dividend process is deterministic. Assuming it is, we then have:

$$G(t,T) = \frac{B(t)}{P(t,T)}e^{-\int_t^T d_s(s)ds}E^Q\left[\frac{S(T)e^{\int_t^T d_s(s)ds}}{B(T)}\middle|F_t\right]$$

and thus, using (16), we obtain the *general cash-and-carry relationship:*

$$G(t,T) = \frac{S(t)e^{-\int_t^T d_s(s)ds}}{P(t,T)} \tag{17}$$

- Equation (17) takes on a particular form when the underlying asset of the forward contract is a *spot exchange rate*. In this case, it is *immaterial* whether domestic and foreign interest rates are deterministic or stochastic, as we will prove. Denoting by S(t) the exchange rate from the domestic viewpoint, i.e. the number of domestic monetary units per foreign monetary unit (say 1,21 US dollars exchanged for 1 Euro), and by $P^f(t,T)$ the foreign zero-coupon bond, the (T-t)-forward exchange rate is equal to:

$$G(t,T) = \frac{S(t)P^f(t,T)}{P(t,T)} \tag{17'}$$

To prove this, we make use of the *principle of international valuation*. According to the latter, in absence of arbitrage opportunity, the present value in domestic currency terms, $V_t^d[.]$, of a future payoff denominated in foreign exchange rate S(T) must be equal to the present value in foreign currency terms of that future payoff, $V_t^f\left[X^f(T)\right]$, times the current spot exchange rate S(t):

$$V_t^d\left[S(T)X^f(T)\right] = S(t)V_t^f\left[X^f(T)\right]$$

Now, consider a future certain payoff of 1 foreign currency unit ($X^f(T)$ =1). We have:

$$G(t,T) = \frac{V_t^d[S(T)1]}{P(t,T)} = S(t)\frac{V_t^f[1]}{P(t,T)} = S(t)\frac{P^f(t,T)}{P(t,T)}$$

which is the desired result, since the present value of 1 foreign currency unit received at T is obviously $P^f(t,T)$.

Equation (17') implies that if the level of domestic interest rates is lower (higher) than that of foreign rates, the forward exchange rate is smaller (larger) than the spot rate, since then P(t,T) is larger (smaller) than $P^f(t,T)$.

2.3. FUTURES PRICES

- Let H(t, T) be price of the futures contract of maturity T written on the spot asset S(t). Assume that the contract is marked to market on a continuous (rather than daily) time basis. We know that in absence of frictions and arbitrage opportunities H(T, T) = S(T). For a moment, assume also that there are no dividends paid out by the underlying asset.

To derive its current price H(t, T), consider the following general strategy: invest initially an amount X(0) = H(0,T) in the riskless asset (in lieu of investing S(0) in the underlying asset); trade at each date t ($0 \leq t \leq T$) $\Delta_H(t)$ units of the (infinitely divisible) contract and re-invest (algebraically) all the margins continuously generated by the futures contracts at the riskless rate. Then the margin account value at each time t, until maturity T, is equal to:

$$X(t) = X(0)\exp\left[\int_0^t r(s)ds\right] + \int_0^t \exp\left[\int_s^t r(s)ds\right]\Delta_H(s)dH(s,T)$$

Applying Ito's lemma to X(t) yields:

$$dX(t) = r(t)X(t)dt + \Delta_H(t)dH(t,T)$$

Now consider a particular strategy on the futures contracts such that:

$$\Delta_H(t) = \exp\left[\int_0^t r(s)ds\right]$$

The margin account dynamics will then be:

$$dX(t) = r(t)X(t)dt + \exp\left[\int_0^t r(s)ds\right]dH(t,T)$$

Applying Ito's lemma yields:

$$X(t) = H(t,T)\exp\left[\int_0^t r(s)ds\right]$$

Consequently,

$$X(T) = S(T)\exp\left[\int_0^T r(s)ds\right]$$

and X(0) is the price at time 0 of this payoff. Since X(0) = H(0,T), we have:

$$H(0,T) = E^Q\left[\frac{S(T)\exp\left[\int_0^T r(s)ds\right]}{\exp\left[\int_0^T r(s)ds\right]}\middle| F_0\right] = E^Q\left[S(T)\middle| F_0\right]$$

and for any date t (< T):

$$H(t,T) = E^Q\left[S(T)\middle| F_t\right] \tag{18}$$

Because of the continuous marking-to-market mechanism, the value of the futures contract is always zero. Therefore, it may seem rather intuitive that its price is a Q-martingale.

- When interest rates are deterministic, we have:

$$H(t,T) = E^Q\left[S(T)\middle| F_t\right] = B(T)E^Q\left[\frac{S(T)}{B(T)}\middle| F_t\right]$$

and, since then P(t,T) = B(t)/B(T) in absence of arbitrage, we get:

$$H(t,T) = B(T)\frac{S(t)}{B(t)} = \frac{S(t)}{P(t,T)} = G(t,T) \tag{19}$$

Under such interest rates, the forward and futures prices are *equal* and given by the *cash-and-carry relationship*. Without dividends, the latter is given by formula (13). With deterministic dividends, the relevant expression is (17).

- Under stochastic interest rates, comparing equations (15) and (18) makes it clear why current forward and futures prices differ: both are the expected value of the underlying asset at date T, S(T), but computed under two different measures: Q^T for the forward and Q for the futures.

To establish the relationship between the two prices, let us compute the drift of the forward price process, μ_G, under the risk-neutral probability Q. Applying Ito's lemma to the cash-and-carry relation (13), we obtain the drift:

$$\begin{aligned}\mu_G &= \mu_S - \mu_P(t,T) + \Sigma_P(t,T)\left(\Sigma_P(t,T) - \Sigma_S\right)' \\ &= \Sigma_P(t,T)\left(\Sigma_P(t,T) - \Sigma_S\right)' \\ &= -\Sigma_P(t,T)\Sigma_G(t,T)'\end{aligned}$$

where the second equality comes from the fact that, under Q, the drifts of the stock price and the bond price processes are equal to the riskless rate r(t), and the third equality uses again Ito's lemma applied to (13) for the diffusion parameters.

Therefore, under Q, the drift of G(t,T) is nothing but the covariance between the forward price relative changes and its underlying bond price relative changes.

We then can use the following well-known theorem. Consider a positive Itô process X(t), satisfying Novikov's condition $E^Q\left[\exp\frac{1}{2}\int_t^T \sigma_X(s)^2 ds \middle| F_t\right] < \infty$ such that, under the measure Q:

$$dX(t) = \mu_X(.)X(t)dt + \sigma_X(.)X(t)d\hat{Z}(t)$$

Then

$$X(t) = E_Q\left[X(T)\exp\left(-\int_t^T \mu_X(s)ds\right)\middle| F_t\right]$$

[<u>Proof</u>: Let $Y(t) = X(T)\exp\left(-\int_t^T \mu_X(s)ds\right)$. By Itô's lemma,

$$\frac{dY(t)}{Y(t)} = \frac{dX(t)}{X(t)} - \mu_X(t)dt = \sigma_X(.)d\hat{Z}(t)$$

Therefore, since Novikov's condition is satisfied, Y(t) is a Q-martingale, and we have:

$$Y(t) = E^Q\left[Y(T)\middle| F_t\right]$$

Hence:

$$X(t)\exp\left(-\int_0^t \mu_X(s)ds\right) = E^Q\left[X(T)\exp\left(-\int_0^T \mu_X(s)ds\right)\middle| F_t\right]$$

which yields the result.]

Consequently, since G(T,T) = S(T), we can write:

$$G(t,T) = E^Q\left[S(T)\exp\left[\int_t^T \mathrm{cov}\left(\frac{dG(u,T)}{G(u,T)},\frac{dP(u,T)}{P(u,T)}\right)du\right]\middle| F_t\right] \tag{20}$$

In the particular case where all variances and covariances are *deterministic*, we then have, using result (18):

$$G(t,T) = H(t,T)\exp\left[\int_t^T \mathrm{cov}\left(\frac{dG(u,T)}{G(u,T)},\frac{dP(u,T)}{P(u,T)}\right)du\right] \tag{21}$$

The forward price thus is larger or smaller than its futures counterpart depending on the sign of the covariance between the forward price changes and the relevant zero-coupon price changes. Obviously, if interest rates are deterministic, this covariance vanishes and we recover G(t,T) = H(t,T). Note for completeness, however, that this assumption is not necessary for the latter equality to hold: it is in fact sufficient that the forward price of the underlying asset and the bond price are statistically independent under Q.

Endnotes

[1] See Heath et al. (1992).
[2] The filtration contains all the events whose probability with respect to P is null. See for instance Karatzas and Shreve (1991).
[3] This equivalence result holds only for simple strategies, i.e. strategies that need a portfolio reallocation only a finite number of times. See Harrison and Kreps (1979) and Harrison and Pliska (1981) for details.
[4] To save space, we do not specify the properties of admissible strategies. See Harrison and Kreps (1979), Harrison and Pliska (1981), Cox and Huang (1989) and Heath et al. (1992).