

# I. Introduction to Homotopy Theory

O.Ya. Viro, D.B. Fuchs

Translated from the Russian  
by C.J. Shaddock

## Contents

Chapter 1. Basic Concepts .....	4
§1. Terminology and Notations .....	4
1.1. Set Theory .....	4
1.2. Logical Equivalence .....	4
1.3. Topological Spaces .....	5
1.4. Operations on Topological Spaces .....	5
1.5. Operations on Pointed Spaces .....	8
§2. Homotopy .....	10
2.1. Homotopies .....	10
2.2. Paths .....	10
2.3. Homotopy as a Path .....	11
2.4. Homotopy Equivalence .....	11
2.5. Retractions .....	11
2.6. Deformation Retractions .....	12
2.7. Relative Homotopies .....	13
2.8. $k$ -connectedness .....	13
2.9. Borsuk Pairs .....	14
2.10. CNRS Spaces .....	15
2.11. Homotopy Properties of Topological Constructions .....	15
2.12. Natural Group Structures on Sets of Homotopy Classes .....	16
§3. Homotopy Groups .....	20
3.1. Absolute Homotopy Groups .....	20

3.2. Digression: Local Systems .....	22
3.3. Local Systems of Homotopy Groups of a Topological Space .....	23
3.4. Relative Homotopy Groups .....	25
3.5. The Homotopy Sequence of a Pair .....	28
3.6. Splitting .....	31
3.7. The Homotopy Sequence of a Triple .....	32
Chapter 2. Bundle Techniques .....	33
§4. Bundles .....	33
4.1. General Definitions .....	33
4.2. Locally Trivial Bundles .....	34
4.3. Serre Bundles .....	36
4.4. Bundles of Spaces of Maps .....	37
§5. Bundles and Homotopy Groups .....	38
5.1. The Local System of Homotopy Groups of the Fibres of a Serre Bundle .....	38
5.2. The Homotopy Sequence of a Serre Bundle .....	39
5.3. Important Special Cases .....	40
§6. The Theory of Coverings .....	41
6.1. Coverings .....	41
6.2. The Group of a Covering .....	42
6.3. Hierarchies of Coverings .....	42
6.4. The Existence of Coverings .....	43
6.5. Automorphisms of a Covering .....	44
6.6. Regular Coverings .....	44
6.7. Covering Maps .....	45
Chapter 3 Cellular Techniques .....	45
§7. Cellular Spaces .....	45
7.1. Basic Concepts .....	45
7.2. Gluing of Cellular Spaces from Balls .....	48
7.3. Examples of Cellular Decompositions .....	49
7.4. Topological Properties of Cellular Spaces .....	52
7.5. Cellular Constructions .....	53
§8. Simplicial Spaces .....	54
8.1. Basic Concepts .....	54
8.2. Simplicial Schemes .....	58
8.3. Simplicial Constructions .....	59
8.4. Stars, Links, Regular Neighbourhoods .....	62
8.5. Simplicial Approximation of a Continuous Map .....	64
§9. Cellular Approximation of Maps and Spaces .....	64
9.1. Cellular Approximation of a Continuous Map .....	64
9.2. Cellular $k$ -connected Pairs .....	65
9.3. Simplicial Approximation of Cellular Spaces .....	66

I. Introduction to Homotopy Theory	3
9.4. Weak Homotopy Equivalence	67
9.5. Cellular Approximation to Topological Spaces	69
9.6. The Covering Homotopy Theorem	71
Chapter 4 The Simplest Calculations	72
§10. The Homotopy Groups of Spheres and Classical Manifolds	72
10.1. Suspension in the Homotopy Groups of Spheres	72
10.2. The Simplest Homotopy Groups of Spheres	73
10.3. The Composition Product	74
10.4. Homotopy Groups of Spheres	75
10.5. Homotopy Groups of Projective Spaces and Lens Spaces	77
10.6. Homotopy Groups of the Classical Groups	78
10.7. Homotopy Groups of Stiefel Manifolds and Spaces	79
10.8. Homotopy Groups of Grassmann Manifolds and Spaces	80
§11. Application of Cellular Techniques	81
11.1. Homotopy Groups of a 1-dimensional Cellular Space	81
11.2. The Effect of Attaching Balls	81
11.3. The Fundamental Group of a Cellular Space	83
11.4. Homotopy Groups of Compact Surfaces	84
11.5. Homotopy Groups of Bouquets	85
11.6. Homotopy Groups of a $k$ -connected Cellular Pair	86
11.7. Spaces with Given Homotopy Groups	87
§12. Appendix	89
12.1. The Whitehead Product	89
12.2. The Homotopy Sequence of a Triad	91
12.3. Homotopy Excision, Quotient and Suspension Theorems	93

# II. Homology and Cohomology

O.Ya. Viro, D.B. Fuchs

Translated from the Russian  
by C.J. Shaddock

## Contents

Chapter 1. Additive Theory .....	98
§1. Algebraic Preparation .....	98
1.1. Complexes and Their Homology .....	98
1.2. Maps and Homotopies .....	99
1.3. Homology sequences .....	100
1.4. The Euler characteristic and the Lefschetz number .....	101
1.5. Change of coefficients .....	103
1.6. Tensor products of complexes and the Künneth formula .....	106
§2. General singular homology theory .....	107
2.1. Basic definitions .....	107
2.2. The simplest calculations .....	110
2.3. Natural transformations; refinement and approximation .....	112
2.4. Excision, factorization, suspension .....	113
2.5. Addition theorems .....	115
2.6. Dependence on the coefficients .....	117
§3. Homology of cellular spaces .....	119
3.1. The cellular complex .....	119
3.2. Interrelations with the singular complex .....	120
3.3. The simplicial case .....	122
3.4. Examples of calculations .....	122
3.5. Other applications .....	123
§4. Homology and homotopy .....	124
4.1. Weak homotopy equivalence and homology .....	124

4.2.	The Hurewicz theorems	124
4.3.	The theorems of Poincaré and Hopf	126
4.4.	Whitehead's theorem	127
4.5.	Some instructive examples	127
§5.	Homology and fixed points	127
5.1.	Lefschetz's theorem	127
5.2.	Smith theory	131
§6.	Other homology and cohomology theories	134
6.1.	The Eilenberg-Steenrod axioms	134
6.2.	An alternative construction of the Eilenberg-Steenrod homology and cohomology theory: the Aleksandrov-Čech theory	136
6.3.	Extraordinary theories	139
6.4.	Homology and cohomology with local coefficients	144
6.5.	Cohomology with coefficients in a sheaf	148
6.6.	Conclusion	152
Chapter 2. Multiplicative theory		152
§7.	Products	152
7.1.	Introduction	152
7.2.	Direct construction of the $\cup$ -product	154
7.3.	Application: the Hopf invariant	155
7.4.	Other products	156
§8.	Homology and manifolds	157
8.1.	Introduction	157
8.2.	The fundamental class	157
8.3.	The Poincaré isomorphisms	159
8.4.	Intersection numbers and Poincaré duality	161
8.5.	Linking coefficients	163
8.6.	Inverse homomorphisms	164
8.7.	The relation with the $\cup$ -product	166
8.8.	Generalizations of the Poincaré isomorphism and duality	167
Chapter 3. Obstructions, characteristic classes and cohomology operations		171
§9.	Obstructions	171
9.1.	Obstructions to extending a continuous map	171
9.2.	The relative case	172
9.3.	Application: cohomology and maps into $K(\pi, n)$ spaces	173
9.4.	Another application: Hopf's theorems	174
9.5.	Obstructions to the extension of sections	175
§10.	Characteristic classes of vector bundles	176
10.1.	Vector bundles	176
10.2.	Associated bundles and characteristic classes	177
10.3.	Characteristic classes and classifying spaces	179
10.4.	The most important properties of Stiefel-Whitney classes	180

10.5. The most important properties of Euler, Chern, and Pontryagin classes .....	182
10.6. Characteristic classes in the topology of smooth manifolds .....	184
§11. Steenrod squares .....	189
11.1. General theory of cohomology operations .....	189
11.2. Steenrod squares and their properties .....	190
11.3. Steenrod squares and Stiefel-Whitney classes .....	191
11.4. Secondary obstructions .....	193
11.5. The non-existence of spheroids with odd Hopf invariant .....	194
References .....	195

# III. Classical Manifolds

D.B. Fuchs

Translated from the Russian  
by the author

## Contents

Introduction . . . . .	199
Chapter 1. Spheres . . . . .	199
§1. Homotopy Groups . . . . .	199
1.1. Generalities . . . . .	199
1.2. Tables and Related Information . . . . .	203
1.3. The Groups $\pi_{n+1}(S^n)$ . . . . .	204
1.4. The Groups $\pi_{n+2}(S^n)$ . . . . .	205
1.5. The Whitehead $J$ -Homomorphism . . . . .	206
§2. Differential Structures . . . . .	207
2.1. Generalities . . . . .	207
2.2. Explicit Constructions of Exotic Spheres . . . . .	208
§3. Appendix . . . . .	209
3.1. Structures . . . . .	209
3.2. Vector Fields and Plane Fields . . . . .	210
3.3. Foliations . . . . .	210
Chapter 2. Lie Groups and Stiefel Manifolds . . . . .	210
§1. Lie Groups: Geometric Information . . . . .	210
1.1. Generalities . . . . .	210
1.2. Some Lie Groups of Low Dimension . . . . .	212
1.3. Homotopy Groups . . . . .	213
§2. Lie Groups: Homological Information . . . . .	215
2.1. Real Cohomology . . . . .	215
2.2. Cohomology Modulo "Good Primes". Integer Cohomology of $U(n)$ and $Sp(n)$ . . . . .	215
2.3. Modulo 2 Cohomology of Orthogonal and Spinor Groups . . . . .	216

2.4. Cohomology of the Exceptional Groups . . . . .	216
2.5. The $K$ -functor . . . . .	217
§3. Stiefel Manifolds . . . . .	217
3.1. Definitions. Geometrical and Homotopical Information . . . . .	217
3.2. Cohomology . . . . .	218
Chapter 3. Grassmann Manifolds and Spaces . . . . .	219
§1. Geometric Information . . . . .	219
1.1. Definitions . . . . .	219
1.2. General Information . . . . .	220
1.3. Embeddings of the Manifolds $G(m, n)$ , $\mathbb{C}G(m, n)$ , $G_+(m, n)$ in Euclidean and Projective Spaces . . . . .	221
§2. Homology Information . . . . .	223
2.1. Cell Decomposition . . . . .	223
2.2. Homology and Cohomology: Cellular Calculations . . . . .	225
2.3. The Cohomology Rings . . . . .	229
2.4. The $K$ -functor . . . . .	232
Chapter 4. Some Other Important Homogeneous Spaces . . . . .	233
§1. Flag Manifolds . . . . .	233
1.1. Generalities . . . . .	233
1.2. Cell Decompositions . . . . .	234
1.3. Homology and Cohomology . . . . .	235
1.4. The Case of Complete Flag Manifolds . . . . .	235
1.5. Generalizations . . . . .	236
§2. The Manifolds $U(n)/SO(n)$ and $U(n)/O(n)$ . . . . .	237
2.1. Generalities . . . . .	237
2.2. Cellular Decompositions . . . . .	238
2.3. Cellular Computation of Homology . . . . .	239
2.4. The Cohomology Rings . . . . .	240
§3. The Manifolds $SO(2n)/U(n)$ and $U(2n)/Sp(n)$ . . . . .	241
Chapter 5. Some Manifolds of Low Dimension . . . . .	242
§1. Closed Surfaces . . . . .	242
1.1. The Standard Surfaces . . . . .	242
1.2. Homotopy Properties . . . . .	243
1.3. Automorphisms . . . . .	244
1.4. Complex Structures . . . . .	245
§2. Some Three-dimensional Manifolds . . . . .	246
2.1. Lens Spaces . . . . .	246
2.2. The Poincaré Sphere . . . . .	248
§3. Some Four-dimensional Manifolds . . . . .	249
References . . . . .	251