2
Numbers

All sorts of number system have been proposed by previous civilizations, but our current system is a positional number system using a base 10. The number 1234 really means the sum of one thousand, plus two hundreds, plus three tens, plus four ones, which can be expressed as $1 \times 1000 + 2 \times 100 + 3 \times 10 + 4 \times 1$. It should be obvious that the base 10 is nothing special, it just so happens that human beings have evolved with 10 digits, which we use for counting. This suggests that any number can be used as a base: 2, 3, 4, 5, 6, 7, etc. In fact, the decimal number system is not very convenient for computer technology, where electronic circuits switch on and off billions of times a second using binary numbers – numbers to a base 2 – with great ease. In this text there is no real need to explore such numbers. This can be left to programmers who have to master number systems such as binary (base 2), octal (base 8) and hexadecimal (base 16).

The only features of numbers we have to revise in this chapter are the families of numbers that exist, what they are used for, and any problems that arise when they are stored in a computer. Let's begin with the natural numbers.

2.1 Natural Numbers

The *natural numbers* $\{0, 1, 2, 3, 4, \ldots\}$ are used for counting, ordering and labelling. Note that negative numbers are not included. We often use natural numbers to subscript a quantity to distinguish one element from another, e.g. $x_1, x_2, x_3, x_4, \ldots$

2.2 Prime Numbers

A natural number that can be divided only by 1 and itself, without leaving a remainder, is called a *prime number*. Examples are $\{2, 3, 5, 7, 11, 13, 17\}$. There are 25 primes less than 100, 168 primes less than 1000 and 455 052 512 primes less than 10 000 000 000. The *fundamental theory of arithmetic* states, 'Any positive integer (other than 1) can be written as the product of prime numbers in one and only one way.' For example, $25 = 5 \times 5; 26 = 2 \times 13; 27 = 3 \times 3 \times 3; 28 = 2 \times 2 \times 7; 29 = 29; 30 = 2 \times 3 \times 5$ and $92\,365 = 5 \times 7 \times 7 \times 13 \times 29$.

In 1742 Christian Goldbach conjectured that every even integer greater than 2 could be written as the sum of two primes:

$$4 = 2 + 2$$
$$14 = 11 + 3$$
$$18 = 11 + 7, \text{etc.}$$

No one has ever found an exception to this conjecture, and no one has ever proved it.

Although prime numbers are enigmatic and have taxed the brains of the greatest mathematicians, unfortunately they play no part in computer graphics!

2.3 Integers

Integers include negative numbers, as follows: $\{\ldots -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$.
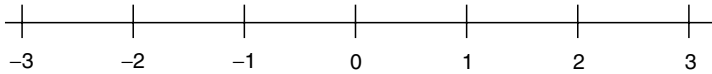
2.4 Rational Numbers

Rational or *fractional* numbers are numbers that can be represented as a fraction. For example, $2, \sqrt{16}, 0.25$ are rational numbers because

$$2 = \frac{4}{2}, \quad \sqrt{16} = 4 = \frac{8}{2}, \quad 0.25 = \frac{1}{4}$$

Some rational numbers can be stored accurately inside a computer, but many others can only be stored approximately. For example, $4/3 = 1.333\,333\ldots$ produces an infinite sequence of threes and has to be truncated when stored as a binary number.

2.5 Irrational Numbers

Irrational numbers cannot be represented as fractions. Examples are $\sqrt{2} = 1.414\,213\,562\ldots, \pi = 3.141\,592\,65\ldots$ and $e = 2.718\,281\,828\ldots$ Such numbers

Fig. 2.1. The number line.

never terminate and are always subject to a small error when stored within a computer.

2.6 Real Numbers

Rational and irrational numbers together comprise the *real numbers*.

2.7 The Number Line

It is convenient to organize numbers in the form of an axis to give them a spatial significance. Figure 2.1 shows such a *number line*, which forms an axis as used in graphs and coordinate systems. The number line also helps us understand complex numbers, which are the 'king' of all numbers.

2.8 Complex Numbers

Leonhard Euler (1707–1783) (whose name rhymes with *boiler*) played a significant role in putting *complex numbers* on the map. His ideas on rotations are also used in computer graphics to locate objects and virtual cameras in space, as we shall see later on.

Complex numbers resolve some awkward problems that arise when we attempt to solve certain types of equations. For example, $x^2 - 4 = 0$ has solutions $x = \pm 2$. But $x^2 + 4 = 0$ has no obvious solutions using real or integer numbers. However, the number line provides a graphical interpretation for a new type of number, the complex number. The name is rather misleading: it is not complex, it is rather simple.

Consider the scenario depicted in Figure 2.2. Any number on the number line is related to the same number with the opposite sign via an anti-clockwise rotation of $180°$. For example, if 3 is rotated $180°$ about zero it becomes -3, and if -2 is rotated $180°$ about zero it becomes 2.

We can now write $-3 = (-1) \times 3$, or $2 = (-1) \times -2$, where -1 is effectively a rotation through $180°$. But a rotation of $180°$ can be interpreted as two consecutive rotations of $90°$, and the question now arises: What does a rotation of $90°$ signify? Well, let's assume that we don't know what the answer is going to be – even though some of you do – we can at least give a name to the operation, and what better name to use than i.
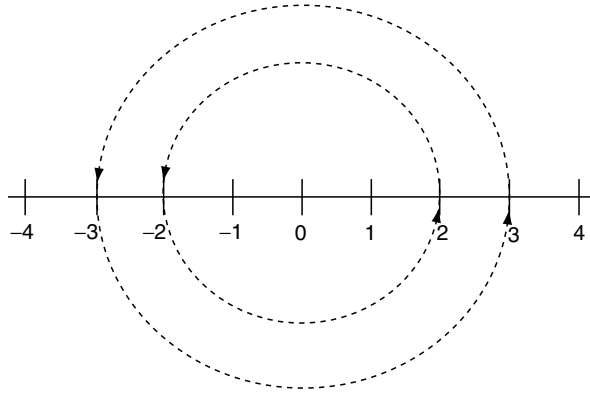
Fig. 2.2. Rotating numbers through 180° reverses their sign.

So the letter i represents an anticlockwise rotation of 90°. Therefore i2 is equivalent to lifting 2 out of the number line, rotating it 90° and leaving it hanging in limbo. But if we take this 'imaginary' number and subject it to a further 90° rotation, i.e. ii2, it becomes -2. Therefore, we can write ii2 $= -2$, which means that ii $= -1$. But if this is so, i $= \sqrt{-1}$!

This gives rise to two types of number: real numbers and complex numbers. Real numbers are the everyday numbers we use for counting and so on, whereas complex numbers have a mixture of real and imaginary components, and help resolve a wide range of mathematical problems.

Figure 2.3 shows how complex numbers are represented: the horizontal number line represents the *real component*, and the vertical number line represents the *imaginary component*.

For example, the complex number $P(1 + i2)$ in Figure 2.3 can be rotated 90° to Q by multiplying it by i. However, we must remember that ii $= -1$:

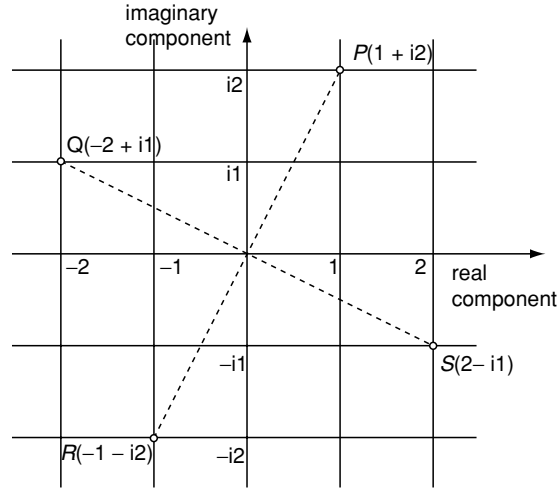$$i(1 + i2) = i1 + ii2$$
$$= i1 - 2$$
$$= -2 + i1$$

$Q(-2 + i1)$ can be rotated another 90° to R by multiplying it by i:

$$i(-2 + i1) = i(-2) + ii1$$
$$= -i2 - 1$$
$$= -1 - i2$$

$R(-1 - i2)$ in turn, can be rotated 90° to S by multiplying it by i:

$$i(-1 - i2) = i(-1) - ii2$$
$$= -i1 + 2$$
$$= 2 - i1$$

Fig. 2.3. The graphical representation of complex numbers.

Finally, $S(2 - i1)$ can be rotated $90°$ to P by multiplying it by i:

$$i(2 - i1) = i2 - ii1$$
$$= i2 + 1$$
$$= 1 + i2$$

Although we rarely use complex numbers in computer graphics, we can see that they are intimately related to Cartesian coordinates, and that the ordered pair $(x, y) \equiv x + iy$.

Before concluding this chapter, I cannot fail to include the famous equation discovered by Euler:

$$e^{i\pi} + 1 = 0 \tag{2.1}$$

which integrates 0, 1, e, π and i in a simple and beautiful arrangement, and is on a par with Einstein's $e = mc^2$.

2.9 Summary

This short chapter made sure that the terminology of numbers was understood, and now provides a good link into the basics of algebra.