

Topology, Quasiperiodic Functions, and the Transport Phenomena

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Summary. In this chapter we give the basic concept of the “topological numbers” in the theory of quasiperiodic functions. Attention is especially paid to appearance of such quantities in transport phenomena, including galvanomagnetic phenomena in normal metals (Sect. 2.1) and the modulations of 2D electron gas (Sect. 2.3). We give a detailed introduction to both these areas and explain in a simple way the appearance of the “integral characteristics” in both these problems. Though this chapter cannot be considered a detailed survey in the area, it explains the main basic features of the corresponding phenomena.

2.1 Introduction

2.1.1 Galvanomagnetic Phenomena in Normal Metals: Classical Results, GSMF Limit

We first consider the transport phenomena connected with the geometry of quasiclassical electron trajectories in the magnetic field \mathbf{B} .

Let us start with the most fundamental case where this kind of phenomena appears in the conductivity of normal metals having complicated Fermi surfaces in the presence of a rather strong magnetic field. This classical part of the solid state physics was started by the Kharkov school of I.M. Lifshitz (Lifshitz, Azbel, Kaganov, Peschansky) in the 1950s and has become an essential part of conductivity theory in normal metals. In particular, they introduced the idea of the geometric strong magnetic field (GSMF) limit. Let us give here some small insight into this area. We start with the classical work of I.M. Lifshitz, M.Ya. Azbel and M.I. Kaganov [1], where the importance of topology of the Fermi surface for the conductivity was established. Namely, the difference between the “simple” Fermi surface (topological “sphere”) (Fig. 2.1a) and more complicated surfaces where the nonclosed quasiclassical electron trajectories can arise was shown. In particular, detailed consideration of the “simple” Fermi surface and surfaces like “warped cylinder” (Fig. 2.1b) for the different directions of \mathbf{B} was made.

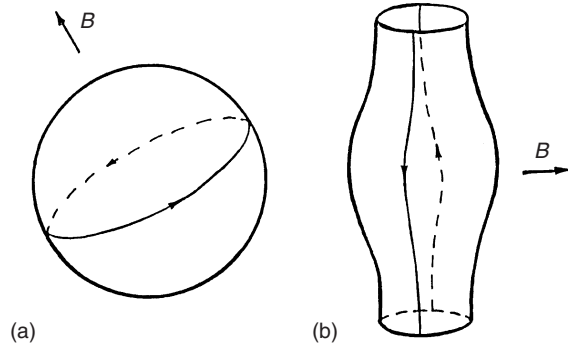


Fig. 2.1. The “simple” Fermi surface having the form of a sphere in the Brillouin zone and the periodic “warped cylinder” extending through an infinite number of Brillouin zones. The quasiclassical electron orbits in \mathbf{p} -space are also shown for a given direction of \mathbf{B}

Figure 2.1 represents the forms of the Fermi surfaces in \mathbf{p} -space and it should be remembered that only one Brillouin zone should be taken into account to get the right phase space volume for the electron states. The values of \mathbf{p} which are different from any reciprocal lattice vector $n_1\mathbf{a}_1 + n_2\mathbf{a}_2 + n_3\mathbf{a}_3$ (where n_i are integers), are physically equivalent to each other and represent the same electron state. The Brillouin zone can then be considered as the parallelogram in the \mathbf{p} -space with the identified opposite sides on the boundary (Fig. 2.2).

Also the Fermi surfaces S_F will then be periodic in \mathbf{p} -space with periods $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$.

Remark. From a topological point of view, we consider the Brillouin zone as the compact three-dimensional torus \mathbb{T}^3 . The corresponding Fermi surfaces will then also be compact surfaces of finite size embedded in \mathbb{T}^3 .

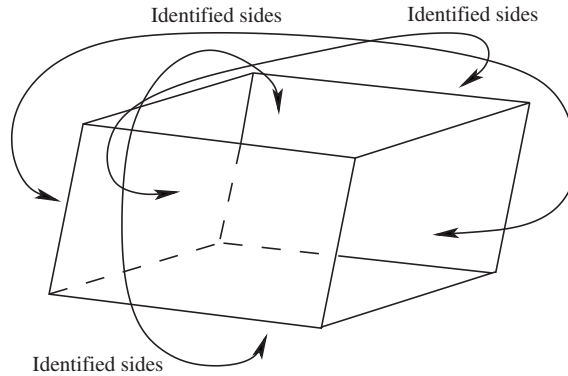


Fig. 2.2. The Brillouin zone in the quasimomentum (\mathbf{p}) space with the identified sides on the boundary

The presence of the homogeneous magnetic field \mathbf{B} generates the evolution of electron states in the \mathbf{p} -space, which can be described by the dynamical system

$$\dot{\mathbf{p}} = \frac{e}{c} [\mathbf{v}_{\text{gr}}(\mathbf{p}) \times \mathbf{B}] = \frac{e}{c} [\nabla\varepsilon(\mathbf{p}) \times \mathbf{B}], \quad (2.1)$$

where $\varepsilon(\mathbf{p})$ is the dependence of energy on the quasimomentum (dispersion relation) and $\mathbf{v}_{\text{gr}}(\mathbf{p}) = \nabla\varepsilon(\mathbf{p})$ is the group velocity at the state \mathbf{p} . Both functions $\varepsilon(\mathbf{p})$ and $\mathbf{v}_{\text{gr}}(\mathbf{p})$ are also periodic functions in \mathbf{p} -space and can be considered as one-valued functions in \mathbb{T}^3 .

System (2.1) has two conservative integrals that are the electron energy and the component of \mathbf{p} along the magnetic field. The electron trajectories can then be represented as the intersections of the constant energy surfaces $\varepsilon(\mathbf{p}) = \text{const.}$ with the planes orthogonal to \mathbf{B} and only the Fermi level $\varepsilon(\mathbf{p}) = \varepsilon_{\text{F}}$ is actually important for the conductivity. It is easy to see then that the global geometry of the “essential” electron trajectories will depend strongly on the form of Fermi surface in \mathbf{p} -space.

Coming back to the Fig. 2.1 we can see that the form of electron trajectories can be quite different for the Fermi surfaces similar to the Fermi surface shown in Fig. 2.1b, we can have periodic nonclosed electron trajectories (if \mathbf{B} is orthogonal to vertical axis), while for the surface on Fig. 2.1a all the trajectories are just closed curves lying in one Brillouin zone for all directions of \mathbf{B} .

We now show that this global geometry plays the main role in the electron motion in the coordinate space also (despite the factorization in \mathbf{p} -space). Thus the electron wave-packet motion in \mathbf{x} -space ($\mathbf{x} = (x, y, z)$) can be found from the additional system

$$\dot{\mathbf{x}} = \mathbf{v}_{\text{gr}}(\mathbf{p}(t)) = \nabla\varepsilon(\mathbf{p}(t))$$

for any trajectory in \mathbf{p} -space after the integration of system (2.1). The structure of system (2.1) permits to claim for example that the xy -projection of “electron motion” in \mathbf{x} -space has the same form as the trajectory in \mathbf{p} -space rotated by $\pi/2$. We can see then that the electron drift in \mathbf{x} -space in magnetic field is also very different for the trajectories shown in Fig. 2.3a, b due to the action of the crystal lattice.

The effect of this “geometrical drift” can be measured experimentally in the rather pure metallic monocrystals if the mean free electron motion time is big enough (such that the electron packet “feels” the geometric features of trajectory between the two scattering acts). The geometric picture requires then that the time between the two scatterings is much longer than the “passing time” through one Brillouin zone for the periodic trajectory and much longer than the “inverse cyclotron frequency” for closed trajectories.¹ For the

¹ This criterion can be actually more complicated for trajectories of more complicated form.

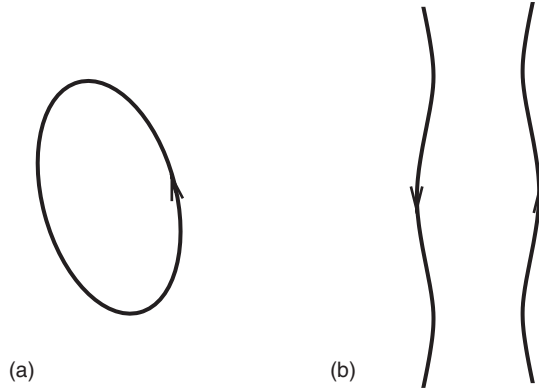


Fig. 2.3. Electron trajectories in \mathbf{p} -space given by the intersections of planes orthogonal to \mathbf{B} for the Fermi surfaces shown in Fig. 2.1a, b for \mathbf{B} orthogonal to vertical axis

approximation of effective mass m^* in crystal this condition can be roughly expressed as $\omega_B \tau \gg 1$, where $\omega_B = eB/m^*c$ is the formal cyclotron frequency and τ is the mean free electron motion time. Let us note that this requirement is satisfied better for big values of B and we consider the formal limit $B \rightarrow \infty$ in this chapter. We call this situation GSMF limit and consider the asymptotic of conductivity tensor for this case.²

We give here the asymptotic form of conductivity tensor obtained in [1] for the case of trajectories shown in Fig. 2.3a, b. Let us take the z -axis in the \mathbf{x} -space along the magnetic field \mathbf{B} . The axes x and y can be chosen arbitrarily for the case of Fig. 2.3a and we take the y -axis along the mean electron drift direction in \mathbf{x} -space for the case of Fig. 2.3b. (It is obvious that the x -axis will then be directed along the mean electron drift in \mathbf{p} -space in this situation). The asymptotic forms of the conductivity tensor can then be written as:

Case 1 (closed trajectories, Fig. 2.3a):

$$\sigma^{ik} \simeq \frac{ne^2\tau}{m^*} \begin{pmatrix} (\omega_B\tau)^{-2} & (\omega_B\tau)^{-1} & (\omega_B\tau)^{-1} \\ (\omega_B\tau)^{-1} & (\omega_B\tau)^{-2} & (\omega_B\tau)^{-1} \\ (\omega_B\tau)^{-1} & (\omega_B\tau)^{-1} & * \end{pmatrix}, \quad \omega_B\tau \gg 1 \quad (2.2)$$

² Formally another condition $\hbar\omega_B \ll \varepsilon_F$ should also be imposed on the magnetic field B . However, this condition is always satisfied for the real metals and all experimentally available magnetic fields (the upper limit is $B \sim 10^3$ – 10^4 T). So we do not pay special attention to this second restriction and assume that the limit $B \rightarrow \infty$ is considered in the “experimental sense,” where the second condition is satisfied.

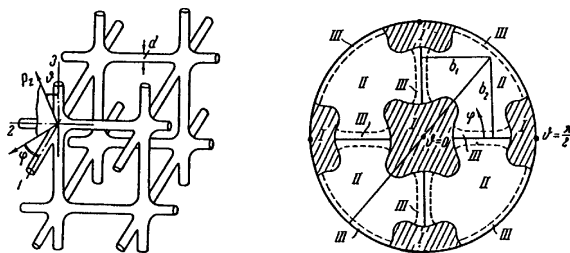


Fig. 2.4. The picture from [2] representing the “thin spatial net” and the corresponding directions of \mathbf{B} on the unit sphere where the nonclosed electron trajectories exist

Case 2 (open periodic trajectories, Fig. 2.3b):

$$\sigma^{ik} \simeq \frac{ne^2\tau}{m^*} \begin{pmatrix} (\omega_B\tau)^{-2} & (\omega_B\tau)^{-1} & (\omega_B\tau)^{-1} \\ (\omega_B\tau)^{-1} & * & * \\ (\omega_B\tau)^{-1} & * & * \end{pmatrix}, \quad \omega_B\tau \gg 1, \quad (2.3)$$

where $*$ indicates some dimensionless constants of the order of 1.

We can see that conductivity reveals the strong anisotropy in the plane orthogonal to \mathbf{B} in the second case, and the mean direction of the electron trajectory in \mathbf{p} -space (not in \mathbf{x}) can be measured experimentally as the zero eigen-direction of σ^{ik} for $B \rightarrow \infty$.

More general types of open electron trajectories are considered in [2, 3]. For example, the open trajectories that are not periodic are found in [2] for the “thin spatial net” (Fig. 2.4a). The open trajectories exist here only for the directions of \mathbf{B} close to main crystallographic axes $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ (Fig. 2.4b). It was shown in [2] that the open trajectories lie in this case in the straight strips of finite width in the plane orthogonal to \mathbf{B} and pass through them. The mean direction of open trajectories is given here by the intersections of plane orthogonal to \mathbf{B} with the main crystallographic planes (xy) , (yz) , and (xz) .

The form of conductivity tensor for this kind of trajectories used in [2] coincides with (2.3).

Some analytical dispersion relations are also considered in [3].³ Let us mention here also the works [4–11] where different experimental (and theoretical) investigations for some real metals were made. Detailed consideration of these results can also be found in the survey articles [12, 13] and the book [14] (see also [15]).

³ Actually this work contains some conceptual mistakes but it also gives some correct features concerning the existence of some open trajectories for these dispersion relations.

2.1.2 Modern Ideas: The GSMF Limit, Topology, and Dynamical Systems

From the physical point of view the problem arising here can be divided into two parts:

(1) The investigation of topology and dynamics of electron trajectories on the Fermi surface. Based on works [16–20] we call this Novikov problem.

(2) The problem of the relation of this dynamics with the physical properties of electric conductivity in the strong magnetic fields (GSMF limit) formulated essentially in the works of Lifshitz group [1–3, 12–14]).

The result of the Lifshitz group is based on the investigation of kinetic equation for the corresponding quasiparticles given in work [1] for the concrete examples. We had to generalize these results, which led us to the formulation of the GSMF limit in the following form: all essential properties of electrical conductivity (under certain restrictions) are determined by the geometry of the dynamical system on the Fermi surface for the limit of large values of \mathbf{B} .

It is worth noting that this part of investigation, including the GSMF-limit principle, was never mathematically rigorously investigated unlike the first part (the Novikov problem) where the investigation was made by the rigorous methods of differential topology. It appeared then that in the case of general position the electron trajectories have the integer topological invariants stable with respect to the small rotations of the magnetic field. These “topological quantum numbers” coincide for different trajectories (i.e., possess the “topological resonance” property). Due to this fact the “Topological quantum numbers” become macroscopic observable quantities. We state that there also exist very interesting cases of the so-called “chaotic trajectories”. This type of trajectories is not yet completely investigated and it seems that new physical phenomena arise there.

Let us now describe in more detail the topological approach to the problem of general classification of all possible electron trajectories regardless the concrete features of the dispersion relation $\varepsilon(\mathbf{p})$ given by Novikov [16] (see also [21–23]). We formulate the Novikov problem here.

Novikov Problem

Let any smooth 3-periodic function $\varepsilon(\mathbf{p})$ be given in the three-dimensional space \mathbb{R}^3 (with arbitrary lattice of periods). Fix any nondegenerate energy level $\varepsilon(\mathbf{p}) = \text{const}$ (i.e., $\nabla\varepsilon(\mathbf{p}) \neq 0$ on this level) and consider the intersections of the corresponding smooth 3-periodic surface by any set of parallel planes in \mathbb{R}^3 . Describe the global geometry of all possible nonsingular (open) trajectories that can arise in the intersections.

The term “the global geometry” means here first the asymptotic behavior of the trajectory when $t \rightarrow \pm\infty$ in the sense of dynamical systems. Let us also formulate here the Novikov conjecture about the generic nonsingular trajectories, which was proved later by his pupils.

Novikov Conjecture

The generic nonsingular open trajectories lie in the straight strips of finite width (in the plane orthogonal to \mathbf{B}) and pass through them.

In the process of proving of Novikov conjecture, the deeper properties of the generic open trajectories were actually revealed. They appeared to be stable with respect to the (small) rotations of the direction of \mathbf{B} . Moreover, it appeared that all the generic open orbits lie on some “warped planes” the quasimomenta space. All these “warped planes” have the integral mean direction (i.e., generated by two reciprocal lattice vectors) and are parallel on average to each other for a given direction of \mathbf{B} . These integral mean directions of “warped planes” appear to be rigid for small rotations of the direction of \mathbf{B} and represent the “Topological quantum numbers” mentioned earlier.

Let us also emphasize that Novikov conjecture is connected with the generic open trajectories and cannot be valid in the special degenerate cases (Tsarev, Dynnikov) as we will see later.

There is also the natural question of what the generic case means in this situation. According to the Novikov conjecture the Hausdorff dimension of the set of directions of \mathbf{B} on the unit sphere where the “nongeneric” open trajectories arise is strictly less than 1 for the generic Fermi surfaces (for some nongeneric Fermi surfaces this dimension can be greater than 1 as for example in the case of the surface $\cos x + \cos y + \cos z = 0$ (see [24, 25])).

Let us now give some historical review on the consideration of the Novikov problem in the topological school (Zorich, Dynnikov, Tsarev), where the basic theorems about the nonclosed trajectories were obtained. We provide here the main breakthroughs in this problem made in [17] (A.V. Zorich) and [20] (Dynnikov).

We first note that even for the rather complicated periodic Fermi surface, the electron trajectories will be quite simple if the direction of \mathbf{B} is purely rational (with respect to reciprocal lattice), i.e., if the plane $\Pi(\mathbf{B})$ orthogonal to \mathbf{B} contains two linearly independent reciprocal lattice vectors. This property can also be formulated in the form where the magnetic fluxes through the faces of elementary cell in the \mathbf{x} -space are proportional to each other with rational coefficients. In this situation the picture arising in $\Pi(\mathbf{B})$ is purely periodic and all open electron trajectories can also be just the periodic curves corresponding precisely to the case (2.3). However, the condition of rationality is completely unstable with respect to any small rotations of \mathbf{B} such that the rational directions give just a set of measure 0 among all the directions of \mathbf{B} .

The remarkable fact proved by Zorich is that the open trajectories reveal the “topologically regular” properties even after the small rotations of the initial purely rational direction. That is, they lie in straight strips of finite width in accordance with the Novikov conjecture (but are not periodic anymore). Let us formulate this in a more precise form.

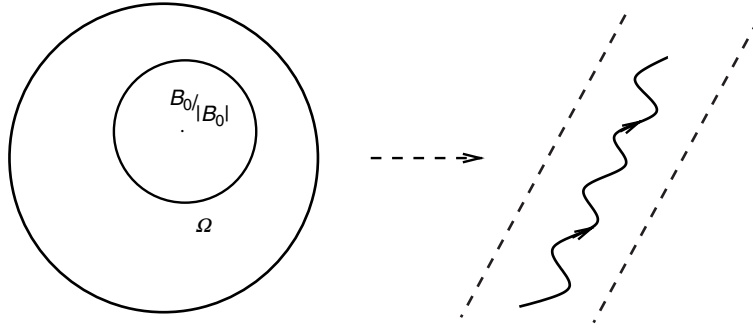


Fig. 2.5. The open region Ω around the purely rational direction \mathbf{B}_0/B_0 on the unit sphere and the general open trajectory lying in the straight strip of finite width in the plane orthogonal to \mathbf{B} for $\mathbf{B}/B \in \Omega$

Theorem 1 (Zorich [17]). *Consider an arbitrary smooth Fermi surface and the rational direction of magnetic field \mathbf{B}_0 such that no singular trajectory connects two different (not equivalent modulo the reciprocal lattice) singular (stagnation) points of the system (2.1). Then there exists a small open region Ω on the unit sphere around direction \mathbf{B}_0 such that all open trajectories (if they exist) lie in straight strips of finite width in the plane orthogonal to \mathbf{B} if $\mathbf{B}/B \in \Omega$ (Fig. 2.5).*

It was also proved by Dynnikov that any trajectory of this kind passes through the corresponding strip and does not come back ([18, 19]).

Let us also mention that the additional topological condition in Theorem 1 has a generic form and generically does not impose anything on the direction \mathbf{B}_0 .

In his theorem, Zorich actually claims that all the rational directions of \mathbf{B} can be extended to some “small open spots” on the unit sphere (parameterizing directions of \mathbf{B}) where we cannot have a situation more complicated than that represented in Fig. 2.5. This set already has the finite measure on the unit sphere and moreover we can conclude that any stable open trajectory can have only the form shown in Fig. 2.5 since the rational directions are dense everywhere on the unit sphere. The Zorich theorem, however, does not permit to state that this situation is the only possible one since the sizes of the “spots” become smaller and smaller for big rational numbers and we cannot claim that they cover all the unit spheres in a general situation.

The next important result was obtained by Dynnikov [20] who proved that the trajectories shown in Fig. 2.5 can be the only stable ones with respect to the small variation of the Fermi energy ε_F for a given dispersion relation $\varepsilon(\mathbf{p})$. We provide the exact form of the Dynnikov theorem in Sect. 2.2 where we will consider this aspect in more detail. We just state here that the methods developed in [20] permitted to prove later that all the cases of open trajectories different from those shown in Fig. 2.5 can appear only “with probability zero”

(i.e., for the directions of \mathbf{B} from the set of measure zero on the unit sphere) for generic Fermi surfaces $S_F : \varepsilon(\mathbf{p}) = \varepsilon_F$ ([24,26]), which gave the final proof of Novikov conjecture for generic open trajectories.

The methods of proving Zorich and Dynnikov theorems gave the basis for the invention of the “topological quantum numbers” introduced in [28] by the present authors (see also the survey articles [29–31]) for conductivity in normal metals. Let us also state that another important property, called the “Topological Resonance,” played a crucial role for physical phenomena in [28]. The main point of this property can be formulated as follows: all the trajectories having the form shown in Fig. 2.5 have the same mean direction in all the planes orthogonal to \mathbf{B} for the generic directions of \mathbf{B} (actually for any not purely rational direction of \mathbf{B}) and give the same form (2.3) of contribution to conductivity tensor in the same coordinate system. This important fact makes experimentally observable the integer-valued topological characteristics of the Fermi surface having the form of the integral planes of reciprocal lattice and corresponding “stability zones” on the unit sphere. We describe in detail these quantities in Sect. 2.2 of our paper. Our goal here is to give the main features of the corresponding picture, so we do not give all the details of the classification of all open trajectories for general Fermi surfaces. However, the picture we will describe serves as the “basic description” of conductivity phenomena and all the other possibilities can be considered as special additional features for the nongeneric directions of \mathbf{B} . Let us also state here that the final classification of open trajectories for generic Fermi surfaces was completed in general by Dynnikov in [27], which solves primarily the Novikov problem. The physical phenomena connected with different types of open trajectories can be found in detail in the survey articles [30,31].

2.1.3 Transport in 2D Electron Gas and Topology of Quasiperiodic Functions

Let us now mention a few words about the so-called generalized Novikov problem in connection with the quasiperiodic functions on the plane with N quasiperiods. According to the standard definition the quasiperiodic function in \mathbb{R}^m with N quasiperiods ($N \geq m$) is a restriction of a periodic function in \mathbb{R}^N (with N periods) to any plane $\mathbb{R}^m \subset \mathbb{R}^N$ of dimension m linearly embedded in \mathbb{R}^N . In our situation we will always have $m = 2$ and the quasiperiodic functions on the plane will be the restrictions of the periodic functions in \mathbb{R}^N to some 2D plane.

Generalized Novikov Problem

Describe the global geometry of open level curves of quasiperiodic function $f(\mathbf{r})$ on the plane with N quasiperiods.

It is easy to see that the generalized Novikov problem gives the Novikov problem for the electron trajectories if we put $N = 3$. Indeed, all the

trajectories in the planes orthogonal to \mathbf{B} can be considered as the level curves of quasiperiodic functions $\varepsilon(\mathbf{p})|_{\Pi(\mathbf{B})}$ with three quasiperiods. As mentioned earlier, the general Novikov problem is solved primarily for $N = 3$. However, the case $N > 3$ becomes very complicated from the topological point of view and no general classification in this case exists at the moment. The only topological result existing now for the general Novikov problem is the analog of Zorich theorem (Theorem 1) for the case $N = 4$ [32] and the general situation is still under investigation.

In Section 2.3 we consider the applications of generalized Novikov problem connected with the “superlattice potentials” for the two-dimensional electron gas in the presence of orthogonal magnetic field. This kind of potentials is connected with modern techniques of “handmade” modulations of 2D electron gas such as the holographic illumination, “gate modulation”, piezoelectric effect, etc. All such modulations are usually periodic in the plane and in many situations the level curves play an important role for the transport phenomena in such systems. The most important thing for us will be the conductivity phenomena in these 2D structures in the presence of orthogonal magnetic field \mathbf{B} . According to the quasiclassical approach the cyclotron electron orbits drift along the level curves of modulation potential in the magnetic field, which gives the “drift contribution” to conductivity in the plane. Among the works devoted to this approach we would like to mention here the article [33], where this method was introduced for the explanation of “commensurability oscillations” of conductivity in potential modulated just in one direction, and [34] where the same approach was used for the explanation of suppression of these oscillations by the second orthogonal modulation in the periodic case. Let us add that all these phenomena correspond to the long free electron motion time, which will now play the role of the “geometric limit” (not $B \rightarrow \infty$) in the second situation.

We will show that the generalized Novikov problem can also arise naturally in these structures if we consider the independent superposition of different periodic modulations. It can be proved that in this case we always obtain the quasiperiodic functions where the number of quasiperiods depends on the complexity of total modulation. The results in Novikov problem can then help to predict the form of the “drift conductivity” in the limit of long free electron motion time. In Sect. 2.3 we give the main features of the situation of superposition of several “1D modulations” where the potentials with a small number of quasiperiods can arise. The detailed consideration of this situation can be found in [35]. However, the Novikov problem also arises in a much more general case of arbitrary superpositions of more complicated (but periodic) structures.

Finally, we would like to mention that the quasiperiodic functions with a large number of quasiperiods can be a model for the random potentials on the plane. The corresponding Novikov problem arises in the percolation theory for such potentials. We will also discuss this situation at the end of Sect. 2.3.

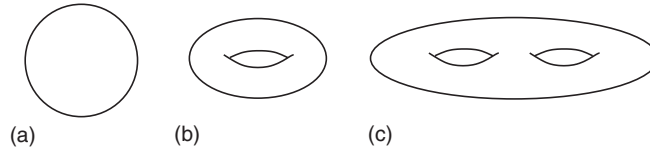


Fig. 2.6. The abstract surfaces (a), (b), and (c) with genera 0, 1, and 2, respectively

2.2 The Classification of Fermi Surfaces and the “Topological Quantum Numbers”

Let us start with the definitions of genus and topological rank of the Fermi surface.

Definition 1. *Let us consider the phase space $\mathbb{T}^3 = \mathbb{R}^3/L$ introduced earlier. After the identification, every component of the Fermi surface becomes the smooth orientable two-dimensional surface embedded in \mathbb{T}^3 . We can then introduce the standard genus of every component of the Fermi surface $g = 0, 1, 2, \dots$ according to standard topological classification depending on whether this component is a topological sphere, torus, sphere with two holes, etc. (Fig. 2.6).*

Definition 2. *Let us introduce the topological rank r as the characteristic of the embedding of the Fermi surface in \mathbb{T}^3 . It is much more convenient in this case to come back to the total \mathbf{p} -space and consider the connected components of the three-periodic surface in \mathbb{R}^3 .*

(1) *The Fermi surface has Rank 0 if each of its connected component can be bounded by a sphere of finite radius.*

(2) *The Fermi surface has Rank 1 if each of its connected component can be bounded by the periodic cylinder of finite radius and there are components that cannot be bounded by the sphere.*

(3) *The Fermi surface has Rank 2 if each of its connected component that can be bounded by two parallel (integral) planes in \mathbb{R}^3 and there are components that cannot be bounded by a cylinder.*

(4) *The Fermi surface has Rank 3 if it contains components that cannot be bounded by two parallel planes in \mathbb{R}^3 .*

Figure 2.7a, b, c, d represents the pieces of the Fermi surfaces in \mathbb{R}^3 with the topological ranks 0, 1, 2, and 3, respectively. As can be seen the genera of the surfaces represented in Fig. 2.7a, b, c, d are also equal to 0, 1, 2, and 3, respectively. However, the genus and the Topological Rank are not necessary equal to each other in the general situation.

Let us discuss briefly the connection between the genus and the topological rank since this will play a crucial role in further consideration.

It is easy to see that the topological rank of the sphere can be only 0 and the Fermi surface consists in this case of the infinite set of the periodically repeated spheres \mathbb{S}^2 in \mathbb{R}^3 .

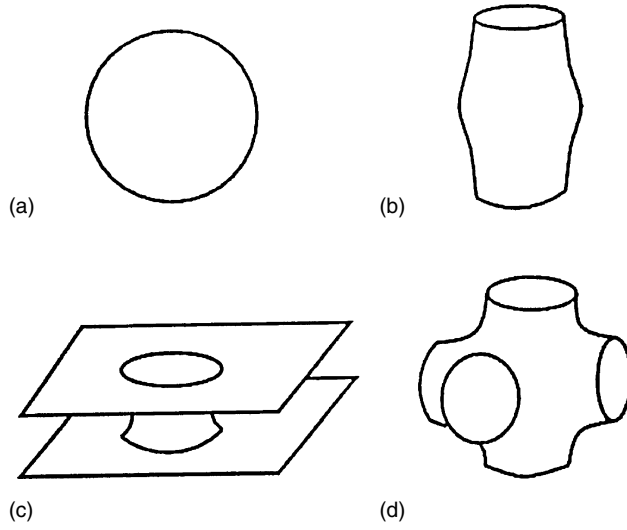


Fig. 2.7. The Fermi surfaces (a), (b), (c), (d) with topological ranks 0, 1, 2, and 3, respectively

The topological rank of the torus \mathbb{T}^2 can take three values $r = 0, 1, 2$. Indeed, it is easy to see that all three cases of periodically repeated tori \mathbb{T}^2 in \mathbb{R}^3 (Rank 0), periodically repeated “warped” integral cylinders (Rank 1), and the periodically repeated “warped” integral planes (Rank 2) give the topological two-dimensional tori \mathbb{T}^2 in \mathbb{T}^3 after the factorization (see Fig. 2.8).

It is not difficult to prove that these are the only possibilities that we can have for embedding of the two-dimensional torus \mathbb{T}^2 in \mathbb{T}^3 . We note here that the mean direction of the “warped periodic cylinder” (embedding of Rank 1) can coincide with any reciprocal lattice vector $n_1\mathbf{a}_1 + n_2\mathbf{a}_2 + n_3\mathbf{a}_3$ in \mathbb{R}^3 . Also the “directions” of the corresponding “warped planes” (embedding of Rank 2)

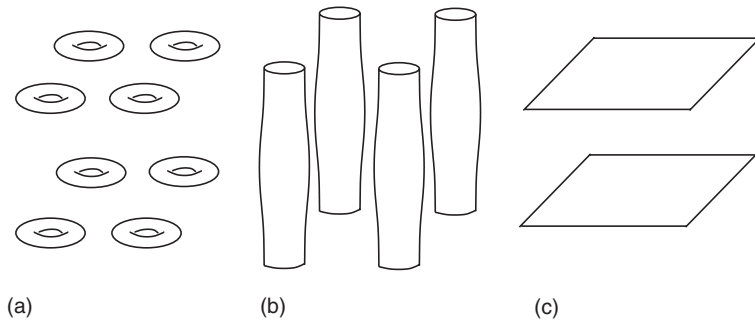


Fig. 2.8. The periodically repeated tori \mathbb{T}^2 , periodically repeated “warped” integral cylinders, and the periodically repeated “warped” integral planes in \mathbb{R}^3

are always generated by two (linearly independent) reciprocal lattice vectors $m_1^{(1)}\mathbf{a}_1 + m_2^{(1)}\mathbf{a}_2 + m_3^{(1)}\mathbf{a}_3$ and $m_1^{(2)}\mathbf{a}_1 + m_2^{(2)}\mathbf{a}_2 + m_3^{(2)}\mathbf{a}_3$. We can thus see that both the embeddings of Rank 1 and Rank 2 of \mathbb{T}^2 in \mathbb{T}^3 are characterized by some integer numbers connected with the reciprocal lattice. Let us also make one more remark about the surfaces of Ranks 0, 1, and 2 in this case. Namely, the case $r = 2$ actually shows one difference from the cases $r = 0$ and 1, which is that the plane in \mathbb{R}^3 is not homologous to 0 in \mathbb{T}^3 (i.e., it does not restrict any domain of “lower energies”) after the factorization. We can conclude that if these planes appear as the connected components of the physical Fermi surface (which is always homologous to 0), they should always come in pairs, Π_+ and Π_- , which are parallel to each other in \mathbb{R}^3 . The factorization of Π_+ and Π_- gives then the two tori $\mathbb{T}_+^2, \mathbb{T}_-^2$ with the opposite homologous classes in \mathbb{T}^3 .

It can be shown that the topological rank of any Fermi surface of genus 2 cannot exceed 2 also. The example of the corresponding embedding of such a component with maximal rank is shown in Fig. 2.7c and represents the two parallel planes connected by cylinders. We will not give the proof of this theorem here but just mention that this fact plays an important role in the classification of nonclosed electron trajectories on the Fermi surface of genus 2. Namely, it can be proved that the open trajectories on the Fermi surface of genus 2 cannot be actually more complicated than the trajectories on the surface of genus 1. In particular they always have the “topologically regular form” in the same way as on the Fermi surface of genus 1 (see Sect. 2.2). Also the same integral characteristics in the cases when this surface has Rank 1 or 2 as in the case of genus 1 can be introduced for genus 2 (actually for any genus if rank is equal to 1 or 2).

Finally we would like to mention that the topological rank of the components with genus $g \geq 3$ can take any value $r = 0, 1, 2, 3$.

Definition 3. *We call the open trajectory topologically regular (corresponding to “topologically integrable” case) if it lies within the straight line of finite width in $\Pi(\mathbf{B})$ and passes through it from $-\infty$ to ∞ . We call all other open trajectories chaotic.*

Let us now discuss the connection between the geometry of the nonsingular electron orbits and the topological properties of the Fermi surface. We briefly consider here the simple cases of Fermi surfaces of Rank 0, 1, and 2 and then come to our basic case of general Fermi surfaces having the maximal rank $r = 3$. We then have the following situations:

- (1) The Fermi surface has topological rank 0.

It is easy to note that in this simplest case all the components of the Fermi surface are compact (Figs. 2.7a, 2.8a) in \mathbb{R}^3 and there are no open trajectories at all.

- (2) The Fermi surface has topological rank 1.

In this case we can have both open and compact electron trajectories. However the open trajectories (if they exist) should be quite simple in this case.

They can arise only if the magnetic field is orthogonal to the mean direction of one of the components of Rank 1 (periodic cylinder) and are periodic with the same integer mean direction (Figs. 2.7b, 2.8b). The corresponding sets of the directions \mathbf{B}/B are just the one-dimensional curves and there cannot be open regions on the unit sphere for which we can find the open trajectories on the Fermi surface.

(3) The Fermi surface has topological rank 2.

It can be easily seen that this case gives much more possibilities for the existence of open orbits for different directions of the magnetic field. In particular, this is the first case where the open orbits can exist for the generic directions of \mathbf{B} . So, in this case we can have the whole regions on the unit sphere such that the open orbits present for any direction of \mathbf{B} belong to the corresponding region. It is easy to see, however, that the open orbits also have quite a simple description in this case. Namely, any open orbit (if it exists) lies in the straight strip of the finite width for any direction of \mathbf{B} not orthogonal to the integral planes given by the components of Rank 2. The boundaries of the corresponding strips in the planes $\Pi(\mathbf{B})$ (orthogonal to \mathbf{B}) will be given by the intersection of $\Pi(\mathbf{B})$ with the pairs of integral planes bounding the corresponding components of Rank 2. It can also be shown [18, 19] that every open orbit passes through the strip from $-\infty$ to $+\infty$ and cannot turn back. We can then see that all the trajectories are “topologically regular” in this case also.

Based on the remarks given earlier, the contribution to the conductivity given by every family of orbits with the same mean direction reveals the strong anisotropy when $\omega_B\tau \rightarrow \infty$ and coincides with the main order with formula (2.3) for the open periodic trajectories.

Trajectories of this type already have all the features of the general topologically integrable situation.

We start now with the most general and complicated case of arbitrary Fermi surface of topological rank 3.

We first describe a convenient procedure [26, 27] of reconstruction of the constant energy surface when the direction of \mathbf{B} is fixed.

We will assume that the system (2.1) has generically only the nondegenerate singularities having the form of the nondegenerate poles or nondegenerate saddle points. The singular trajectories passing through the critical points (and the critical points themselves) divide the set of trajectories into different parts corresponding to different types of trajectories on the Fermi surface. We are not interested here in the geometry of compact electron trajectories in the “geometric limit” $\omega_B\tau \rightarrow \infty$. It is not difficult to show that the pieces of the Fermi surface carrying the compact orbits can be either infinite or finite cylinders in \mathbb{R}^3 bounded by the singular trajectories (some of them may be just points of minimum or maximum) at the bottom and at the top (see Fig. 2.9).

Let us now remove all the parts containing the nonsingular compact trajectories from the Fermi surface. The remaining part,

$$S_F / (\text{compact nonsingular trajectories}) = \cup_j S_j,$$

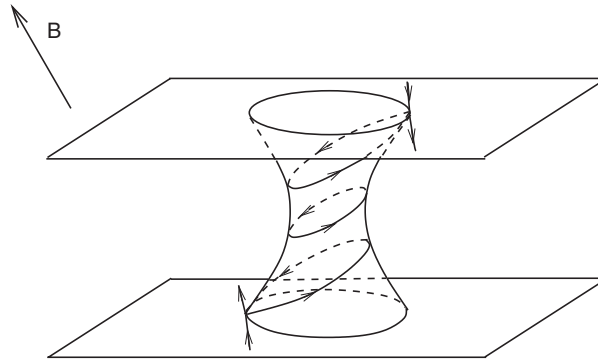


Fig. 2.9. The cylinder of compact trajectories bounded by the singular orbits (the simplest case of just one critical point on the singular trajectory)

is a union of the two manifolds S_j with boundaries ∂S_j , which are the compact singular trajectories. The generic type in this case is a separatrix orbit with just one critical point like in Fig. 2.9.

It is obvious that the open orbit will not be affected at all by the construction described here and the rest of the Fermi surface gives the same open orbits as all possible intersections with different planes orthogonal to \mathbf{B} .

Definition 4. We call every piece S_j the “Carrier of open trajectories.”

Let us fill in the holes by topological 2D discs lying in the planes orthogonal to \mathbf{B} and get the closed surfaces (see Fig. 2.10)

$$\bar{S}_j = S_j \cup (2D \text{ discs}).$$

This procedure again gives the periodic surface \bar{S}_ε after the reconstruction and we can define the “compactified carriers of open trajectories” both in \mathbb{R}^3 and \mathbb{T}^3 .

It is obvious that the reconstructed surface can be used instead of the original Fermi surface for the determination of open trajectories. Let us ask a question: can the reconstructed surface be simpler than the original one?

The answer is positive and moreover it can be proved that “generically” the reconstructed surface consists of components of genus 1 only. This remarkable fact gives the very powerful instrument for the consideration of open trajectories on the arbitrary Fermi surface.

In fact, the proof of Theorem 1 was based on the statement that the genus of every compactified carrier of open orbits \bar{S}_j is equal to 1 in this case.

Let us now formulate the theorem of Dynnikov [20], which made the second main breakthrough in the Novikov problem.

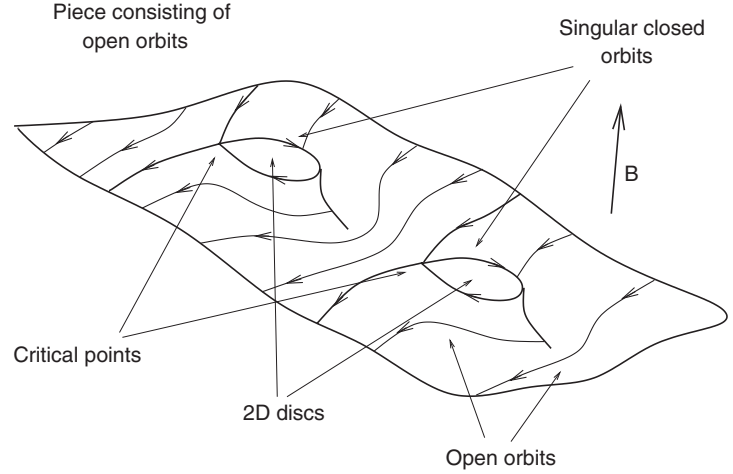


Fig. 2.10. The reconstructed constant energy surface with removed compact trajectories and the two-dimensional discs attached to the singular trajectories in the generic case of just one critical point on every singular trajectory

Theorem 2 (Dybnikov [20]). *Let a generic dispersion relation*

$$\varepsilon(\mathbf{p}) : \mathbb{T}^3 \rightarrow \mathbb{R}$$

be given such that for level $\varepsilon(\mathbf{p}) = \varepsilon_0$ the genus g of some carrier of open trajectories \bar{S}_i is greater than 1. Then there exists an open interval $(\varepsilon_1, \varepsilon_2)$ containing ε_0 such that for all $\varepsilon \neq \varepsilon_0$ in this interval the genus of the carrier of open trajectories is less than g .

Theorem 2 claims that only the “topologically integrable case” can be stable with respect to the small variations of energy level also.

The formulated theorems permit us to reduce the consideration of open orbits in any stable situation to the case of the surfaces of genus 1 where the Fermi surface can have topological rank 0, 1, or 2 only. It is easy to see that the Rank 0 cannot appear just by definition of the reconstructed surface \bar{S}_ε since it can contain only the compact trajectories. Rank 1 is possible in \bar{S}_ε only for special directions of \mathbf{B} . Indeed, the component of Rank 1 has the mean integral direction in \mathbb{R}^3 and can contain the open (periodic) trajectories only if \mathbf{B} is orthogonal to this integral vector in \mathbf{p} -space. The corresponding open trajectories are thus not absolutely stable with respect to the small rotations of \mathbf{B} and cannot exist for the open region on the unit sphere.

We can then claim that the only generic situation for \bar{S}_ε is a set of components of Rank 2, which are the periodic warped planes in this case. The corresponding electron trajectories can then belong just to “Topologically integrable” case being the intersections of planes orthogonal to \mathbf{B} with the periodically deformed planes in the \mathbf{p} -space.

An important property of the compactified components of genus 1 arising for the generic directions of \mathbf{B} is the following: they are all parallel on average to \mathbb{R}^3 and do not intersect each other. This property mentioned in [28] and called later the “topological resonance” plays an important role in the physical phenomena connected with geometry of open trajectories. In particular, all the stable topologically regular open trajectories in all planes orthogonal to \mathbf{B} have the same mean direction and give the same form (2.3) of contribution to conductivity in the appropriate coordinate system common for all of them. This fact gives the experimental possibility of measuring the mean direction of noncompact topologically regular orbits both in \mathbf{x} and \mathbf{p} spaces from the anisotropy of conductivity tensor σ^{ik} .

We reiterate that the surface \overline{S}_ε is the abstract construction depending on the direction of \mathbf{B} and does not exist a priori in the Fermi surface S_{ε_F} . The important fact, however, is the stability of the surface \overline{S}_ε with respect to the small rotations of \mathbf{B} . This means in particular that the common direction of the components of Rank 2 is locally stable with respect to the small rotations of \mathbf{B} , which can then be found from the conductivity experiments. From the physical point of view, all the regions on the unit sphere where the stable open orbits exist can be represented as the “stability zones” Ω_α such that each zone corresponds to some integral plane Γ_α common to all the points of stability zone Ω_α . The plane Γ_α is then the integral plane in reciprocal lattice, which defines the mean directions of open orbits in \mathbf{p} -space for any direction of \mathbf{B} belonging to Ω_α just as the intersection with the plane orthogonal to \mathbf{B} . As can be easily seen from the form of (2.3), this direction always coincides with the unique direction in \mathbb{R}^3 corresponding to the decrease of conductivity as $\omega_B \tau \rightarrow \infty$.

The corresponding integral planes Γ_α can then be given by three integer numbers $(n_\alpha^1, n_\alpha^2, n_\alpha^3)$ (up to the common multiplier) from the equation

$$n_\alpha^1[\mathbf{x}]_1 + n_\alpha^2[\mathbf{x}]_2 + n_\alpha^3[\mathbf{x}]_3 = 0,$$

where $[\mathbf{x}]_i$ are the coordinates on the basis $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ of the reciprocal lattice, or equivalently

$$n_\alpha^1(\mathbf{x}, \mathbf{l}_1) + n_\alpha^2(\mathbf{x}, \mathbf{l}_2) + n_\alpha^3(\mathbf{x}, \mathbf{l}_3) = 0,$$

where $\{\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3\}$ is the basis of the initial lattice in the coordinate space.

We see then that the direction of conductivity decreasing $\hat{\eta} = (\eta_1, \eta_2, \eta_3)$ satisfies the relation

$$n_\alpha^1(\hat{\eta}, \mathbf{l}_1) + n_\alpha^2(\hat{\eta}, \mathbf{l}_2) + n_\alpha^3(\hat{\eta}, \mathbf{l}_3) = 0$$

for all the points of stability zone Ω_α , which makes possible the experimental observation of numbers $(n_\alpha^1, n_\alpha^2, n_\alpha^3)$.

The numbers $(n_\alpha^1, n_\alpha^2, n_\alpha^3)$ are called in [28] the “topological quantum numbers” of a dispersion relation in metal.

We can now consider the result of [2] about the “thin spatial net” as a particular case of this general theorem where the integer planes take the

simplest possibility of being the main planes xy , yz , xz . If we now introduce the “topological quantum numbers” for this situation, we will have only the triples $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, and $(0, 0, \pm 1)$ for this Fermi surface.

In general, we can state that the unit sphere should be divided into (open) parts where the open orbits are absent on the Fermi level for given directions of \mathbf{B} and “stability zones” Ω_α where the open orbits exist on the Fermi level and have “topologically regular” form. Each stability zone corresponds to the triple of “topological quantum numbers” giving the integral direction of periodically deformed two-dimensional planes in $\bar{S}_{\varepsilon_F}(\mathbf{B})$, which are swept by the zero eigen-vector of σ^{ik} for $\mathbf{B} \in \Omega_\alpha$.

We now state that the “topologically regular” trajectories are generic open trajectories, nonetheless they are not ideal for rather complicated Fermi surfaces. Namely, for rather complicated Fermi surfaces and the special directions of \mathbf{B} , the chaotic cases can also arise (Tsarev, Dynnikov).

It was first shown by Tsarev [36] that the more complicated chaotic open orbits can still exist on rather complicated Fermi surfaces S_F . An example of an open trajectory that does not lie in any finite strip of finite width was constructed. However, the trajectory had in this case the asymptotic direction of not even being restricted by any straight strip of finite width in the plane orthogonal to \mathbf{B} . The corresponding asymptotic behavior of conductivity should also reveal the strong anisotropy properties in the plane orthogonal to \mathbf{B} although the exact form of σ^{ik} will be slightly different from (2.3) for this type of trajectories. For the same reason, the asymptotic direction of orbit can be measured experimentally in this case.

The more complicated examples of chaotic open orbits were constructed in [26] for the Fermi surface having genus 3. These types of open orbits do not have any asymptotic direction in the planes orthogonal to \mathbf{B} and have a rather complicated form of “walking everywhere” in these planes.

The corresponding contribution to σ^{ik} is also very different for this kind of trajectories [37]. In particular, it appears that this contribution becomes 0 in all the directions including the direction of \mathbf{B} for $B \rightarrow \infty$. The total conductivity tensor σ^{ik} has then only the contribution of compact electron trajectories in the conductivity along \mathbf{B} , which does not disappear when $B \rightarrow \infty$. The corresponding effect can be observed experimentally as the local minima of the longitudinal (i.e., parallel to \mathbf{B}) conductivity for the points of the unit sphere where this kind of trajectories can appear. A more detailed description of σ^{ik} in this case can be found in [37].

Let us add that Dynnikov proved recently that the measure of chaotic cases on the unit sphere is 0 for generic Fermi surfaces [26, 27]. The systematic investigation of the open orbits was completed in general after the works [17, 20, 26, 28] in [27]. In particular the total picture of different types of the open orbits for generic dispersion relations was presented. Let us formulate here the main results of [27] in the form of a Theorem.

Theorem 3 (Dybnikov [27]). *Let us fix the dispersion relation $\varepsilon = \varepsilon(\mathbf{p})$ and the direction of \mathbf{B} of irrationality 3 and consider all the energy levels for $\varepsilon_{\min} \leq \varepsilon \leq \varepsilon_{\max}$. Then:*

(1) *The open electron trajectories exist for all the energy values ε belonging to the closed connected energy interval $\varepsilon_1(\mathbf{B}) \leq \varepsilon \leq \varepsilon_2(\mathbf{B})$, which can degenerate to just one energy level $\varepsilon_1(\mathbf{B}) = \varepsilon_2(\mathbf{B}) = \varepsilon_0(\mathbf{B})$.*

(2) *For the case of the nontrivial energy interval the set of compactified carriers of open trajectories \bar{S}_ε is always a disjoint union of two-dimensional tori \mathbb{T}^2 in \mathbb{T}^3 for all $\varepsilon_1(\mathbf{B}) \leq \varepsilon \leq \varepsilon_2(\mathbf{B})$. All the tori \mathbb{T}^2 for all the energy levels do not intersect each other and have the same (up to the sign) indivisible homology class $c \in H_2(\mathbb{T}^3, \mathbb{Z})$, $c \neq 0$. The number of tori \mathbb{T}^2 is even for every fixed energy level and the corresponding covering \bar{S}_ε in \mathbb{R}^3 is a locally stable family of parallel (“warped”) integral planes $\Pi_i^2 \subset \mathbb{R}^3$ with common direction given by c . The form of \bar{S}_ε described here is locally stable with the same homology class $c \in H_2(\mathbb{T}^3)$ under small rotations of \mathbf{B} . All the open electron trajectories at all the energy levels lie in the strips of finite width with the same direction and pass through them. The mean direction of the trajectories is given by the intersections of planes $\Pi(\mathbf{B})$ with the integral family Π_i^2 for the corresponding “stability zone” on the unit sphere.*

(3) *The functions $\varepsilon_1(\mathbf{B})$, $\varepsilon_2(\mathbf{B})$ defined for the directions of \mathbf{B} of irrationality 3 can be continued on the unit sphere S^2 as the piecewise smooth functions such that $\varepsilon_1(\mathbf{B}) \geq \varepsilon_2(\mathbf{B})$ everywhere on the unit sphere.*

(4) *For the case of trivial energy interval $\varepsilon_1 = \varepsilon_2 = \varepsilon_0$ the corresponding open trajectories may be chaotic. The carrier of the chaotic open trajectory is homologous to 0 in $H_2(\mathbb{T}^3, \mathbb{Z})$ and has genus ≥ 3 . For the generic energy level $\varepsilon = \varepsilon_0$ the corresponding directions of magnetic fields belong to the countable union of the codimension 1 subsets. Therefore a measure of this set is equal to 0 on S^2 .*

We give here the results connected with generic directions of \mathbf{B} and do not consider the special cases when \mathbf{B} is purely or “partly” rational. The corresponding effects are actually simpler than formulated earlier and can be easily added to this general picture. Survey articles [27, 29–31] provide all the details (both from mathematical and physical point of view).

2.3 Quasiperiodic Modulations of 2D Electron Gas and the Generalized Novikov Problem

In this section we provide a general description about the quasiperiodic modulations of 2D electron gas and the main topological aspects for the special class of such structures. Let us first discuss about different modern modulation techniques and the quasiclassical electron behavior in such systems.

We first point here the holographic illumination of high-mobility 2D electron structures (AlGaA–GaAs heterojunctions) at temperatures $T \leq 4.2$ K (see, for example, [38]). In these experiments the expanded laser beam was split into two parts, which gave an interference picture with the period a on the 2D sample. The illumination caused the additional ionization of atoms near the 2D junction, which remained for a rather long period of time after the illumination. During this relaxation time, the additional periodic potential $V(\mathbf{r}) = V(x)$, $V(x) = V(x + a)$ arose in the plane and the electron behavior was determined by the orthogonal magnetic field \mathbf{B} and the potential $V(x)$.

The quasiclassical consideration for the case $|V(x)| \ll \varepsilon_F$ was first considered by Beenakker [33] for the explanation of “commensurability oscillations” in such structures found in [38]. According to this approach the quasiclassical electrons near the Fermi level move around the cyclotron orbits in the magnetic field and drift due to potential $V(x)$ in the plane. Since only the electrons near Fermi level ε_F play the main role in conductivity, we can introduce the characteristic cyclotron radius $r_B = m^*v_F/eB$ for the Fermi velocity v_F . The corresponding drift of the electron orbits near the Fermi level will then be determined by the averaged effective potential $V_B^{\text{eff}}(x)$ given by the averaging of $V(\mathbf{r}) = V(x)$ over the cyclotron orbit with radius r_B centered at the point \mathbf{r} (Fig. 2.11).

The potential $V_B^{\text{eff}}(x)$ is different from $V(x)$ but has the same symmetry and also depends only on x . The drift of the cyclotron orbits is along the level curves of $V_B^{\text{eff}}(x)$, which are very simple in this case (just the straight lines along the y -axis) and the corresponding velocity v_{drift} is proportional to the absolute value of gradient $|V_B^{\text{eff}}(x)|$ at each level curve. The analytic dependence of $|V_B^{\text{eff}}(x)|$ on the value of B (based on the commensurability of $2r_B$ with the (integer number) $\times a$) was used in [33] for the explanation of the oscillations of conductivity along the fringes with the value of B .

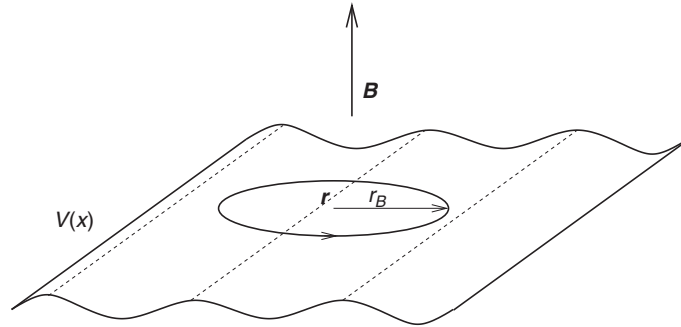


Fig. 2.11. The averaging of the the potential $V(x)$ over the cyclotron orbit with radius r_B centered at the point \mathbf{r}

In article [34] the situation with the double-modulated potentials made by the superposition of two interference pictures was also considered. The corresponding potential $V(\mathbf{r})$ is double periodic in \mathbb{R}^2 in this case and the same is true for potentials $V_B^{\text{eff}}(\mathbf{r})$. The consideration used the same quasiclassical approach for the potential $V_B^{\text{eff}}(\mathbf{r})$ based on the analysis of its level curves. It was then shown in [34] that the second modulation should suppress the commensurability oscillations in this case, which disappear completely for the equal intensities of two (orthogonal) interference pictures.

It is also obvious that all the open drift trajectories can be only periodic in the case of periodic $V_B^{\text{eff}}(\mathbf{r})$.

It seems that the situation with the quasiperiodic modulations of 2D electron gas did not appear in experiments. However, we think that this situation is also very natural for the technique described earlier and can be considered from the point of view of the generalized Novikov problem. The corresponding approach was developed in [35] for the special cases of superpositions of several (three and four) interference pictures on the plane. Nonetheless, as we already mentioned, the Novikov problem also arises actually for any picture given by superposition of several periodic pictures in the plane. The corresponding potentials can have many quasiperiods in this case and the Novikov problem can then reveal much more complicated (chaotic) properties than described in [35].

We next describe here just the main points of “topologically regular” behavior in the case of the superpositions of three and four interference pictures, which give the quasiperiodic potentials $V(\mathbf{r})$ and $V_B^{\text{eff}}(\mathbf{r})$ with three and four quasiperiods on the plane. Unlike the previous works we do not pay much attention to the analytic dependence on B and investigate mainly the geometric properties of conductivity in this situation.

Before we start the geometric consideration, we wish to also state that the holographic illumination is not a unique way of producing the superlattice potentials for the two-dimensional electron gas. Let us mention here the works [39–49] where the different techniques using the biasing of the specially made metallic gates and the piezoelectric effect were considered. Both 1D and 2D modulated potentials as well as more general periodic potentials with square and hexagonal geometry appeared in this situation. Actually these techniques give much more possibilities to produce the potentials of different types with the quasiperiodic properties.

Let us now have three independent interference pictures on the plane with three different generic directions of fringes η_1, η_2, η_3 and periods a_1, a_2, a_3 (see Fig. 2.12).

The total intensity $I(\mathbf{r})$ will be the sum of intensities

$$I(\mathbf{r}) = I_1(\mathbf{r}) + I_2(\mathbf{r}) + I_3(\mathbf{r})$$

of the independent interference pictures.

We assume that there are at least two noncoinciding directions (say η_1, η_2) among the set (η_1, η_2, η_3) .

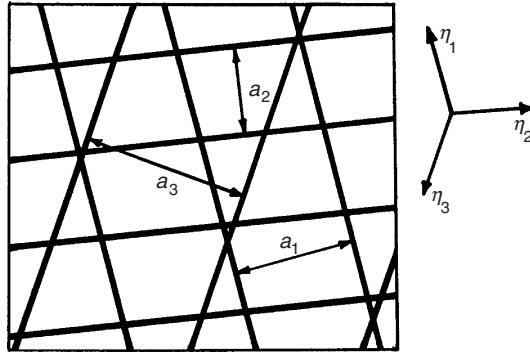


Fig. 2.12. The schematic sketch of the three independent interference pictures on the plane with different periods and intensities

It can be shown that the potentials $V(\mathbf{r})$ and $V_{\mathbb{B}}^{\text{eff}}(\mathbf{r})$ can be represented in this situation as the quasiperiodic functions with three quasiperiods in the plane.

Let us now introduce the important definition of the “quasiperiodic group” acting on the potentials described earlier.

Definition 5. *Let us fix the directions η_1, η_2, η_3 and periods a_1, a_2, a_3 of the interference fringes in Fig. 2.12 and consider all independent parallel shifts of positions of different interference pictures in \mathbb{R}^2 . All the potentials $V'(\mathbf{r})$ (and the corresponding $V_{\mathbb{B}}^{\text{eff}}(\mathbf{r})$) made in this way are related by the transformations of a quasiperiodic group.*

According to the definition the quasiperiodic group is a three-parametric Abelian group isomorphic to the three-dimensional torus \mathbb{T}^3 due to the periodicity of every interference picture.⁴

We state that potential $V(\mathbf{r})$ is generic if it has no periods in \mathbb{R}^2 , is periodic if it has two linearly independent periods in \mathbb{R}^2 , and is “partly periodic” if it has just one (up to the integer multiplier) period in \mathbb{R}^2 .

It can also be shown that the quasiperiodic group does not change the “periodicity” of potentials $V(\mathbf{r})$, $V_{\mathbb{B}}^{\text{eff}}(\mathbf{r})$.

The results for the Novikov problem can also be applied in this situation. We formulate here the main results for the generic potentials $V(\mathbf{r})$ (the special additional features can be found in [35]). Let us formulate here the theorem from [35] about the drift trajectories for the generic potentials of this kind based on the topological theorems for Novikov problem in 3-dimensional case (formulated earlier).

⁴ It is obvious that the quasiperiodic group contains the ordinary translations as the algebraic subgroup.

Theorem 4 [35]. *Let us fix the value of B and consider the generic quasiperiodic potential $V_B^{\text{eff}}(\mathbf{r})$ made by three interference pictures and taking the values in some interval $\varepsilon_{\min}(B) \leq V_B^{\text{eff}}(\mathbf{r}) \leq \varepsilon_{\max}(B)$. Then:*

(1) *Open quasiclassical trajectories $V_B^{\text{eff}}(\mathbf{r}) = c$ always exist either in the connected energy interval*

$$\varepsilon_1(B) \leq c \leq \varepsilon_2(B)$$

($\varepsilon_{\min}(B) < \varepsilon_1(B) < \varepsilon_2(B) < \varepsilon_{\max}(B)$) or just at one energy value $c = \varepsilon_0(B)$.

(2) *For the case of the finite interval ($\varepsilon_1(B) < \varepsilon_2(B)$) all the nonsingular open trajectories correspond to topologically regular case, i.e., lie in the straight strips of the finite width and pass through them. All the strips have the same mean directions for all the energy levels $c \in [\varepsilon_1(B), \varepsilon_2(B)]$ such that all the open trajectories are on average parallel to each other for all values of c .*

(3) *The values $\varepsilon_1(B)$, $\varepsilon_2(B)$, or $\varepsilon_0(B)$ are the same for all the generic potentials connected by the “quasiperiodic group.”*

(4) *For the case of the finite energy interval ($\varepsilon_1(B) < \varepsilon_2(B)$) all the nonsingular open trajectories also have the same mean direction for all the generic potentials connected by the “quasiperiodic group” transformations.*

We again see that the “topologically regular” open trajectories are also generic for this situation as seen earlier.

Let us now consider the asymptotic behavior of conductivity tensor when $\tau \rightarrow \infty$ (mean free electron motion time). We consider here only the “topologically regular” case. Let us point out that the full conductivity tensor can be represented as the sum of two terms

$$\sigma_0^{ik}(B) = \sigma_0^{ik}(B) + \Delta\sigma^{ik}(B).$$

In the approximation of the drifting cyclotron orbits, the parts $\sigma_0^{ik}(B)$ and $\Delta\sigma^{ik}(B)$ can be interpreted as caused by the (infinitesimally small) difference in the electron distribution function on the same cyclotron orbit (weak angular dependence) and the (infinitesimally small) difference in the occupation of different trajectories by the centers of cyclotron orbits at different points of \mathbb{R}^2 (on the same energy level) as the linear response to the (infinitesimally) small external field \mathbf{E} , respectively.

The first part $\sigma_0^{ik}(B)$ has the standard asymptotic form:

$$\sigma_0^{ik}(B) \sim \frac{ne^2\tau}{m^{\text{eff}}} \begin{pmatrix} (\omega_B\tau)^{-2} & (\omega_B\tau)^{-1} \\ (\omega_B\tau)^{-1} & (\omega_B\tau)^{-2} \end{pmatrix}$$

for $\omega_B\tau \gg 1$ due to the weak angular dependence ($\sim 1/\omega_B\tau$) of the distribution function on the same cyclotron orbit. We then have that the corresponding longitudinal conductivity decreases for $\tau \rightarrow \infty$ in all the directions in \mathbb{R}^2 and the corresponding condition is just $\omega_B\tau \gg 1$ in this case.

For the part $\Delta\sigma^{ik}(B)$ the limit $\tau \rightarrow \infty$ should, however, be considered as the condition that every trajectory is passed for a rather long time by the

drifting cyclotron orbits to reveal its global geometry. Thus another parameter τ/τ_0 , where τ_0 is the characteristic time of completion of close trajectories, should be used in this case and we should put the condition $\tau/\tau_0 \gg 1$ to have the asymptotic regime for $\Delta\sigma^{ik}(B)$. In this situation the difference between the open and closed trajectories plays the main role, and the asymptotic behavior of conductivity can be calculated in the form analogous to that used in [1–3] for the case of normal metals. That is:

$$\Delta\sigma^{ik}(B) \sim \frac{ne^2\tau}{m^{\text{eff}}} \begin{pmatrix} (\tau_0/\tau)^2 & \tau_0/\tau \\ \tau_0/\tau & (\tau_0/\tau)^2 \end{pmatrix}$$

in the case of closed trajectories and

$$\Delta\sigma^{ik}(B) \sim \frac{ne^2\tau}{m^{\text{eff}}} \begin{pmatrix} * & \tau_0/\tau \\ \tau_0/\tau & (\tau_0/\tau)^2 \end{pmatrix}$$

(* ~ 1) for the case of open topologically regular trajectories if the x -axis coincides with the mean direction of trajectories.

The condition $\tau/\tau_0 \gg 1$ is much stronger than $\omega_B\tau \gg 1$ in the situation described here according to the definition of the slow drift of the cyclotron orbits. We can keep then just this condition in our further considerations and assume that the main part of conductivity is given by $\Delta\sigma^{ik}(B)$ in this limit. It is also obvious that the magnetic field B should not be “very strong” in this case.

Based on these remarks, we can now write the main part of the conductivity tensor $\sigma^{ik}(B)$ in the limit $\tau \rightarrow \infty$ for the case of topologically regular open orbits. Let us take the x -axis along the mean direction of open orbits and the y -axis orthogonal to x . The asymptotic form of σ^{ik} , $i, k = 1, 2$ can then be written as:

$$\sigma^{ik} \sim \frac{ne^2\tau}{m^{\text{eff}}} \begin{pmatrix} * & \tau_0/\tau \\ \tau_0/\tau & (\tau_0/\tau)^2 \end{pmatrix}, \quad \tau_0/\tau \rightarrow 0, \quad (2.4)$$

where $*$ is some value of the order of 1 (constant as $\tau_0/\tau \rightarrow 0$).

The asymptotic form of σ^{ik} makes possible the experimental observation of the mean direction of topologically regular open trajectories if the value τ/τ_0 is rather big.

Let us now introduce the “topological numbers” characterizing the regular open trajectories analogous to those introduced in [28] for the case of normal metals. We will first give the topological definition of these numbers using the action of the “quasiperiodic group” on the quasiperiodic potentials [35].

We assume that we have the “topologically integrable” situation where the topologically regular open trajectories exist in some finite energy interval $\varepsilon_1(B) \leq c \leq \varepsilon_2(B)$. According to Theorem 4 the values $\varepsilon_1(B)$, $\varepsilon_2(B)$ and the mean directions of open trajectories are the same for all the potentials constructed from our potential with the aid of the “quasiperiodic group.” It also follows from the topological picture that all the topologically regular

trajectories are absolutely stable under the action of the “quasiperiodic group” for the generic $V_B^{\text{eff}}(\mathbf{r})$ and can just “crawl” in the plane for the continuous action of such transformations.

We take the first interference picture (η_1, a_1) and shift continuously the interference fringes in the direction orthogonal to η_1 to the distance a_1 keeping two other pictures unchanged. At the end we will have the same potentials $V(x, y)$ and $V_B^{\text{eff}}(x, y)$ due to the periodicity of the first interference picture with period a_1 . Let us fix now some energy level $c \in (\varepsilon_1(B), \varepsilon_2(B))$ and look at the evolution of nonsingular open trajectories (for $V_B^{\text{eff}}(x, y)$) while making our transformation. We know that we should have the parallel open trajectories in the plane each time and the initial picture should coincide with the final according to the construction. The form of trajectories can change during the process but their mean direction will be the same according to Theorem 4 (“topological resonance”).

We can then claim that every open trajectory will be “shifted” to another open trajectory of the same picture by our continuous transformation. It is not difficult to prove that all the trajectories will then be shifted by the same number of positions n_1 (positive or negative), which depends on the potential $V_B^{\text{eff}}(x, y)$ (Fig. 2.13).

The number n_1 is always even since all the trajectories appear by pairs with the opposite drift directions.

Let us now do the same with the second and the third sets of the interference fringes and get an integer triple (n_1, n_2, n_3) , which is a topological characteristic of potential $V_B^{\text{eff}}(x, y)$ (the “positive” direction of the numeration of trajectories should be the same for all these transformations).

The triple (n_1, n_2, n_3) can be represented as:

$$(n_1, n_2, n_3) = M(m_1, m_2, m_3),$$

where $M \in \mathbb{Z}$ and (m_1, m_2, m_3) is the indivisible integer triple.

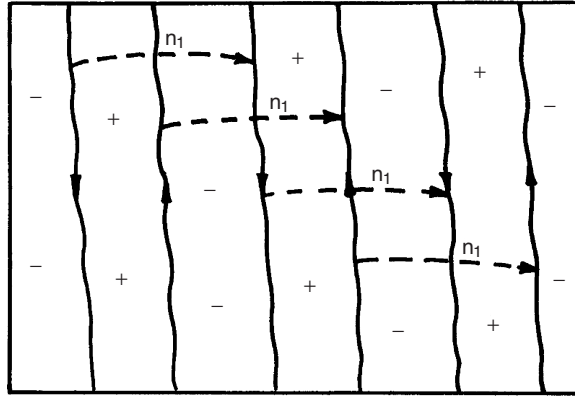


Fig. 2.13. The shift of “topologically regular” trajectories by a continuous transformation generated by the special path in the “quasiperiodic group”

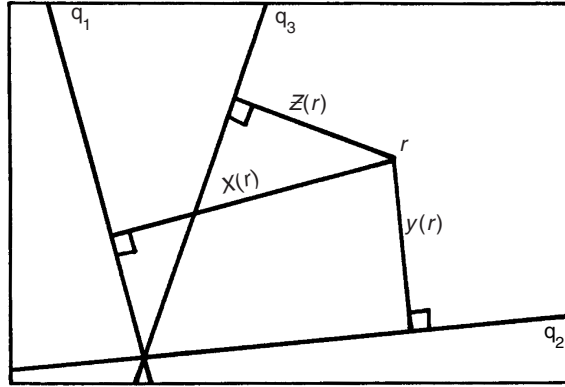


Fig. 2.14. The functions $X(\mathbf{r})$, $Y(\mathbf{r})$, and $Z(\mathbf{r})$ on the plane

The numbers (m_1, m_2, m_3) (defined up to the common sign) play now the role of “topological numbers” for this situation. For direct experimental observation of these numbers, the connection between these numbers and the mean direction of the “topologically regular” trajectories can play an important role. This connection is described as follows.

Let us draw three straight lines q_1, q_2, q_3 with the directions η_1, η_2, η_3 (Fig. 2.12) and choose the “positive” and “negative” half-planes for every line q_i on the plane. Let us now consider three linear functions $X(\mathbf{r}), Y(\mathbf{r}), Z(\mathbf{r})$ on the plane that are the distances from the point \mathbf{r} to the lines q_1, q_2, q_3 with the signs “+” or “-” depending on the half-plane for the corresponding line q_i (Fig. 2.14). Let us choose here the signs “+” or “-” such that the gradients of $X(\mathbf{r}), Y(\mathbf{r}), Z(\mathbf{r})$ coincide with directions of shifts of the corresponding interference pictures in the definition of (m_1, m_2, m_3) .

Theorem 5 [35]. Consider the functions

$$X'(\mathbf{r}) = X(\mathbf{r})/a_1, \quad Y'(\mathbf{r}) = Y(\mathbf{r})/a_2, \quad Z'(\mathbf{r}) = Z(\mathbf{r})/a_3$$

in \mathbb{R}^2 . The mean direction of the regular open trajectories is given by the linear equation:

$$m_1 X'(x, y) + m_2 Y'(x, y) + m_3 Z'(x, y) = 0, \quad (2.5)$$

where (m_1, m_2, m_3) is the indivisible integer triple introduced earlier.

Let us now describe the situation with four independent sets of interference fringes in the plane (see also [35]). In general we get here the quasiperiodic potentials $V(\mathbf{r}), V_{\mathbb{B}}^{\text{eff}}(\mathbf{r})$ with four quasiperiods. The situation in this case is more complicated than in the case $N = 3$ and no general classification of open trajectories exists at the time. At the moment only the theorem analogous to Zorich result can be formulated in this situation [32]. According to the Novikov theorem we can claim that the “small perturbations” of purely

periodic potentials having four quasiperiods have the “topologically regular” level curves like in the previous case.

The purely periodic potentials $V(\mathbf{r})$ give the same dense set in the space of parameters $\eta_1, \eta_2, \eta_3, \eta_4, a_1, a_2, a_3, a_4$ and can be found in any small open region of this space. The Novikov theorem claims then that every potential of this kind can be surrounded by the “small open ball” in the space of parameters $\eta_1, \eta_2, \eta_3, \eta_4, a_1, a_2, a_3, a_4$ where the open level curves will always demonstrate the “topologically regular” behavior. The set of potentials thus obtained has finite measure among all potentials and the “topologically regular” open trajectories can be found with finite probability also in this case. However, we do not claim here that the chaotic behavior has measure 0 for four quasiperiods and moreover we also expect the nonzero probability for the chaotic trajectories in this more complicated case.

The topologically regular cases demonstrate here the same “regularity properties” as in the previous case including the “Topological numbers.” Thus, we can introduce in the same way the action of the quasiperiodic group on the space of potentials with four quasiperiods and define in the same way the four tuples (m_1, m_2, m_3, m_4) of integer numbers characterizing the topologically regular cases in this situation.

Also, the analogous theorem about mean directions of the regular trajectories can be formulated in this case. Namely, if we introduce the functions $X(\mathbf{r}), Y(\mathbf{r}), Z(\mathbf{r}), W(\mathbf{r})$ in the same way as for the case of three quasiperiods (above) and the corresponding functions

$$X'(\mathbf{r}) = X(\mathbf{r})/a_1, \dots, W'(\mathbf{r}) = W(\mathbf{r})/a_4,$$

we can write the equation for the mean direction of open trajectories on the plane in the form:

$$m_1 X'(\mathbf{r}) + m_2 Y'(\mathbf{r}) + m_3 Z'(\mathbf{r}) + m_4 W'(\mathbf{r}) = 0.$$

The numbers (m_1, m_2, m_3, m_4) are stable with respect to the small variations of $\eta_1, \eta_2, \eta_3, \eta_4, a_1, a_2, a_3, a_4$ (and the intensities of the interference pictures I_1, I_2, I_3, I_4) and correspond again to some “stability zones” in this space of parameters.

A brief mention is now made about the limit of Novikov problem for large values of N . The following problem can be formulated as:

Give a description of global geometry of the open level curves of quasiperiodic function $V(\mathbf{r})$ in the limit of large numbers of quasiperiods.

We can claim that the open level curves should exist here also in the connected energy interval $[\varepsilon_1, \varepsilon_2]$ on the energy scale, which can degenerate just to one point ε_0 .⁵ We expect that the “topologically regular” open trajectories can also exist in this case. However the probability of “chaotic behavior” should increase for the cases of large N , which is closer now to random potential situation. The corresponding behavior can be considered then as the

⁵ The proof given in [24] for the case of 3 quasiperiods works actually for any N .

“percolation problem” in special models of random potentials given by quasi-periodic approximations. Certainly, this model can be quite different from the others. Nevertheless, we expect a similar behavior of the chaotic trajectories for rather big N also in this rather special model. This area, however, is still under investigation.

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