

Contents

Introduction.....	1
Chapter 1. Classical Calculus of Variations	5
§1. Euler Equation.....	5
1.1. Brachistochrone Problem	5
1.2. Euler Equation	6
1.3. Geodesics on a Riemannian Manifold	9
§2. Hamiltonian Formalism	12
2.1. Legendre Transform	12
2.2. Canonical Variables	15
2.3. Mechanical Meaning of the Canonical Variables	16
2.4. Variation Formula for a Functional with Movable Endpoints ..	17
2.5. Transversality Conditions in the Problem with Movable Endpoints	18
2.6. Weierstrass–Erdmann Conditions	20
2.7. Hamilton–Jacobi Equation	23
§3. Theory of the Second Variation	25
3.1. Problem of the Second Variation	25
3.2. Legendre Necessary Condition	26
3.3. The Associated Problem and the Definition of a Conjugate Point	28
3.4. Necessary Conditions for the Positive Semidefiniteness of $\delta^2 J$.	29
§4. Riccati Equation	30
4.1. Sufficient Conditions for the Positive Definiteness of $\delta^2 J$	30
§5. Morse Index	35
§6. Jacobi Envelope Theorem	38
§7. Strong Minimum	42
7.1. Weierstrass Necessary Condition	42
§8. Poincaré–Cartan Integral Invariant	45
8.1. Exterior Differential Forms	45
8.2. Poincaré–Cartan Integral Invariant	47
8.3. Legendre Manifolds	50
§9. Fields of Extremals	53
9.1. Hilbert Invariant Integral	53
9.2. Embedding an Extremal in a Field and Focal Points	55
Chapter 2. Riccati Equation in the Classical Calculus of Variations	60
§1. Riccati Equation as a Sufficient Condition for Positivity of the Second Variation	60
§2. Riccati Equation for a Problem with Differential Constraints	62
2.1. Problem with Differential Constraints	63

2.2.	Optimal Control Problem	63
2.3.	Linear-Quadratic Problem	64
2.4.	Bellman Equation	66
§3.	Riccati Equation and the Grassmann Manifold	68
3.1.	Grassmann Manifold	69
3.2.	Riccati Equation as a Flow on the Grassmann Manifold	70
§4.	Grassmann Manifolds of Lower Dimension	72
4.1.	Quaternions	73
4.2.	Homotopic Paths	77
Chapter 3. Lie Groups and Lie Algebras		80
§1.	Lie Groups: Definition and Examples	80
§2.	Lie Algebras	84
2.1.	Vector Fields on a Manifold	85
2.2.	Lie Algebras	87
§3.	Lie Groups of Lower Dimension	88
3.1.	Topological Structure of the Groups $SO(3)$ and $Spin(3)$	88
3.2.	Topological Structure of the Group $SL(2, \mathbb{R})$	90
3.3.	Topological Structure of the Groups $Sp(1, \mathbb{R})$, $U(1)$, and $SU(2)$	90
§4.	Adjoint Representation and Killing Form	91
4.1.	Adjoint Representation	91
4.2.	Killing Form	93
4.3.	Subalgebras and Ideals	93
§5.	Semisimple Lie Groups	96
5.1.	Compact Lie Algebras	98
§6.	Homogeneous and Symmetrical Spaces	100
6.1.	Symmetrical Spaces	101
§7.	Totally Geodesic Submanifolds	107
7.1.	Lie Group Isometries	107
7.2.	Geodesics in the Quotient Space of Lie Groups	110
Chapter 4. Grassmann Manifolds		112
§1.	Three Approaches to the Description of the Grassmann Manifolds	112
1.1.	Local Coordinates on the Grassmann Manifold	112
1.2.	Invariant Description of Grassmann Manifolds	114
1.3.	Metric on the Grassmann Manifold	114
1.4.	Grassmann Manifolds as Symmetrical Spaces	114
1.5.	Plücker Embeddings	115
§2.	Lagrange–Grassmann Manifolds	117
2.1.	Coordinates on a Lagrange–Grassmann Manifold	118
2.2.	Lagrange–Grassmann Manifold as a Homogeneous Space	119
2.3.	The Manifold $A(\mathbb{R}^{2n})$ as a Symmetrical Space	123
§3.	Riccati Equation as a Flow on the Manifold $G_n(\mathbb{R}^{2n})$	123

§4. Systems Associated with a Linear System of Differential Equations .	126
4.1. Associated Systems on Grassmann Manifolds	128
Chapter 5. Matrix Double Ratio	130
§1. Matrix Double Ratio on the Grassmann Manifold	130
§2. Clifford Algebras	135
§3. Totally Geodesic Submanifolds of Grassmann Manifolds	137
§4. Curves with a Scalar Double Ratio	143
§5. Fourth Harmonic as a Geodesic Symmetry	147
5.1. Manifold of Isotropic Planes	148
§6. Clifford Parallels	150
§7. Connection Between Clifford Parallels and Isoclinic Planes	154
§8. Matrix Double Ratio on the Lagrange–Grassmann Manifold	155
§9. Morse–Maslov–Arnol’d Index in the Leray–Kashivara Form	158
§10. Fourth Harmonic as an Isometry of the Lagrange–Grassmann Manifold	162
§11. Application of the Matrix Double Ratio to the Study of the Riccati Equation	162
Chapter 6. Complex Riccati Equations	166
§1. Cartan–Siegel Domains	166
§2. Klein–Poincaré Upper Half-Plane and Generalized Siegel Upper Half-Plane	174
2.1. Generalized Siegel Upper Half-Plane	177
2.2. Siegel Half-Plane as a Symmetrical Space	178
2.3. Action of $\text{Sp}(n, \mathbb{R})$ on the Boundary of the Siegel Half-Plane . .	184
2.4. Cayley Transform	185
§3. Complexified Riccati Equation as a Flow on the Generalized Siegel Upper Half-Plane	187
§4. Flow on Cartan–Siegel Homogeneity Domains	189
4.1. Riccati-Type Equation for a Linear System Whose Matrix Belongs to a Given Lie Algebra	190
4.2. Flow on the Siegel Homogeneity Domain of Type I	192
4.3. Flow on the Siegel Homogeneity Domain of Type II	194
4.4. Flow on the Siegel Homogeneity Domain of Type IV	196
§5. Matrix Analog of the Schwarz Differential Operator	198
5.1. Classical Schwarz Differential Operator	200
5.2. Schwarz Operator and a Linear Second-Order Differential Equation	202
5.3. Schwarz Operator and the Riccati Equation	203
5.4. Matrix Analog of the Schwarz Operator	205

Chapter 7. Higher-Dimensional Calculus of Variations	208
§1. Minimal Surfaces	208
§2. Necessary Optimality Conditions for a Multiple Integral	212
2.1. Euler Equation	213
2.2. Second Variation	215
2.3. Variational Equation	216
§3. Vector Bundles	217
§4. Distributions and the Frobenius Theorem	219
§5. Connection in a Linear Bundle	227
§6. Levi-Civita Connection	230
6.1. Torsion and Curvature of a Connection of a Vector Bundle	233
§7. Nonnegativity Conditions of the Second Variation	236
§8. Field Theory in the Weyl Form	240
§9. Caratheodory Transformation	244
9.1. Condition for Realizability of the Caratheodory Transformation	247
§10. Field Theory in the Caratheodory Form	248
 Chapter 8. On the Quadratic System of Partial Differential Equations Related to the Minimization Problem for a Multiple Integral	 254
§1. Riccati Equation in the Case of the Degenerate Legendre Condition	254
§2. Reducing the Dirichlet Integral to the Integral of Its Principal Part	257
§3. Relation of the Riccati Partial Differential Equation to the Euler Equation	261
3.1. Compactification of the Space on Which the Riccati Partial Differential Equation is Defined	263
§4. Connection Defined by a Solution to the Riccati Partial Differential Equation	264
4.1. Potentiality Condition for Tensor Fields	270
 Epilogue	 272
 Appendix to the English Edition	 273
 References	 276
 Index	 282