
Contents

Introduction	ix
1 Arbitrage Pricing Theory: The One-Period Model	1
1.1 The Arrow–Debreu Model	2
1.2 Security-Space Diagram: A Geometric Interpretation of Theorem 1.1	8
1.3 Replication	11
1.4 The Binomial Model	13
1.5 Complete and Incomplete Markets	14
1.6 The One-Period Trinomial Model	16
1.7 Exercises	18
References and Further Reading	19
2 The Binomial Option Pricing Model	21
2.1 Recursion Relation for Pricing Contingent Claims	22
2.2 Delta-Hedging and the Replicating Portfolio	24
2.3 Pricing European Puts and Calls	26
Portfolio Delta	27
Money-Market Account	27
Puts	28
2.4 Relation Between the Parameters of the Tree and the Stock	
Price Fluctuations	28
Calibration of the Volatility Parameter	31
Expected Growth Rate	32
Implementation of Binomial Trees	33
2.5 The Limit for $dt \rightarrow 0$: Log-Normal Approximation	34
2.6 The Black–Scholes Formula	35
References and Further Reading	39
3 Analysis of the Black–Scholes Formula	41
3.1 Delta	42
Option Deltas	44
3.2 Practical Delta Hedging	45
3.3 Gamma: The Convexity Factor	48
3.4 Theta: The Time-Decay Factor	51
3.5 The Binomial Model as a Finite-Difference Scheme for the Black–Scholes Equation	54
References and Further Reading	55

4 Refinements of the Binomial Model	57
4.1 Term-Structure of Interest Rates	57
4.2 Constructing a Risk-Neutral Measure with Time-Dependent Volatility	63
4.3 Deriving a Volatility Term-Structure from Option Market Data	66
4.4 Underlying Assets That Pay Dividends	70
4.5 Futures Contracts as the Underlying Security	73
4.6 Valuation of a Stream of Uncertain Cash Flows	75
References and Further Reading	76
5 American-Style Options, Early Exercise, and Time-Optionality	77
5.1 American-Style Options	77
5.2 Early-Exercise Premium	78
5.3 Pricing American Options Using the Binomial Model: The Dynamic Programming Equation	80
5.4 Hedging	82
5.5 Characterization of the Solution for $dt \ll 1$: Free-Boundary Problem for the Black–Scholes Equation	82
References and Further Reading	88
A A PDE Approach to the Free-Boundary Condition	89
A.1 A Proof of the Free Boundary Condition	90
6 Trinomial Model and Finite-Difference Schemes	93
6.1 Trinomial Model	93
6.2 Stability Analysis	95
6.3 Calibration of the Model	96
6.4 “Tree-Trimming” and Far-Field Boundary Conditions	100
6.5 Implicit Schemes	103
References and Further Reading	106
7 Brownian Motion and Ito Calculus	107
7.1 Brownian Motion	107
7.2 Elementary Properties of Brownian Paths	109
7.3 Stochastic Integrals	111
7.4 Ito’s Lemma	117
7.5 Ito Processes and Ito Calculus	120
References and Further Reading	122
A Properties of the Ito Integral	123
8 Introduction to Exotic Options: Digital and Barrier Options	127
8.1 Digital Options	128
European Digitals	128
American Digitals	135
8.2 Barrier Options	139
Pricing Barrier Options Using Trees or Lattices	141
Closed-Form Solutions	142
Hedging Barrier Options	145
8.3 Double Barrier Options	146
Range Discount Note	147
Range Accruals	148
Double Knock-out Options	150
References and Further Reading	150

A	Proofs of Lemmas 8.1 and 8.2	151
A.1	A Consequence of the Invariance of Brownian Motion Under Reflections	151
A.2	The Case $\mu \neq 0$	153
B	Closed-Form Solutions for Double-Barrier Options	155
B.1	Exit Probabilities of a Brownian Trajectory from a Strip $-B < Z < A$	155
B.2	Applications to Pricing Barrier Options	158
9	Ito Processes, Continuous-Time Martingales, and Girsanov's Theorem	161
9.1	Martingales and Doob–Meyer Decomposition	161
9.2	Exponential Martingales	163
9.3	Girsanov's Theorem	165
	References and Further Reading	168
A	Proof of Equation (9.11)	169
10	Continuous-Time Finance: An Introduction	171
10.1	The Basic Model	171
10.2	Trading Strategies	173
10.3	Arbitrage Pricing Theory	176
	References and Further Reading	181
11	Valuation of Derivative Securities	183
11.1	The General Principle	183
11.2	Black–Scholes Model	185
11.3	Dynamic Hedging and Dynamic Completeness	189
11.4	Fokker–Planck Theory: Computing Expectations Using PDEs	193
	References and Further Reading	196
A	Proof of Proposition 11.5	197
12	Fixed-Income Securities and the Term-Structure of Interest Rates	199
12.1	Bonds	199
12.2	Duration	206
12.3	Term Rates, Forward Rates, and Futures-Implied Rates	209
12.4	Interest-Rate Swaps	212
12.5	Caps and Floors	217
12.6	Swaptions and Bond Options	218
12.7	Instantaneous Forward Rates: Definition	221
12.8	Building an Instantaneous Forward-Rate Curve	224
	References and Further Reading	227
13	The Heath–Jarrow–Morton Theorem and Multidimensional Term-Structure Models	229
13.1	The Heath–Jarrow–Morton Theorem	230
13.2	The Ho–Lee Model	234
13.3	Mean Reversion: The Modified Vasicek or Hull–White Model	237
13.4	Factor Analysis of the Term-Structure	239
13.5	Example: Construction of a Two-Factor Model with Parametric Components	245
13.6	More General Volatility Specifications in the HJM Equation	248
	References and Further Reading	251

14 Exponential-Affine Models	253
14.1 A Characterization of EA Models	255
14.2 Gaussian State-Variables: General Formulas	258
14.3 Gaussian Models: Explicit Formulas	261
14.4 Square-Root Processes and the Non-Central Chi-Squared Distribution	264
14.5 One-Factor Square-Root Model: Discount Factors and Forward Rates	268
References and Further Reading	272
A Behavior of Square-Root Processes for Large Times	273
B Characterization of the Probability Density Function of Square-Root Processes	275
C The Square-Root Diffusion with $\nu = 1$	277
15 Interest-Rate Options	279
15.1 Forward Measures	279
Definition and Examples	279
15.2 Commodity Options with Stochastic Interest Rate	282
15.3 Options on Zero-Coupon Bonds	283
15.4 Money-Market Deposits with Yield Protection	285
Forward Rates and Forward Measures	286
15.5 Pricing Caps	289
General Considerations	289
Cap Pricing with Gaussian Models	292
Cap Pricing with Square-Root Models	293
Cap Pricing and Implied Volatilities	297
15.6 Bond Options and Swaptions	299
General Pricing Relations	299
Jamshidian's Theorem	301
Volatility Analysis	303
15.7 Epilogue: The Brace–Gatarek–Musiela model	308
References and Further Reading	312
Index	313