Table of Contents

Part One. Algebraic Equations

Chapter 1. Three Muslimic Authors 3

Part A. Al-Khwārizmī 3

1. The Man and his Work 3
2. Al-jabr and al-muqabala 4
3. On Mensuration 5
4. On the Jewish Calendar 7
5. On Legacies 7
6. The Solution of Quadratic Equations 7
7. The Geography 9
8. On Hindu Numerals 9
9. The Astronomical Tables 9
10. The "Sindhind" 10
11. The "Method of the Persians" 11
12. Al-Khwārizmī's Sources 13

Part B. Tabit ben Qurra 15

The Sabians 15
The Life of Tabit ben Qurra 16
On the Motion of the Eighth Sphere 17
Geometrical Verification of the Solution of Quadratic Equations . . . 18
On Amicable Numbers 21

Part C. Omar Khayyam 24

The Algebra of Omar Khayyam 24
Omar Khayyam on Ratios 29

Chapter 2. Algebra in Italy 32

Part A. From Leonardo da Pisa to Luca Pacioli 32

The Connection Between Trade and Civilization in Medieval Italy . . 32
Life and Work of Fibonacci 33
1. The "Liber Abbaci" 35
2. The "Practica geometriae" 39
3. The Book "Flos" 40
4. The Letter to Theodorus 40
5. The "Liber quadratorum" 40

Three Florentine Abbacists 42
1. Maestro Benedetto 42
2. Maestro Biaggio 43
3. Antonio Mazzinghi 44
Two Anonymous Manuscripts 45
Luca Pacioli 46
Part B. Master Dardi of Pisa 47
Part C. The Solution of Cubic and Biquadratic Equations 52
Scipione del Ferro 52
Tartaglia and Cardano 54
Lodovico Ferrari 56
Rafael Bombelli 59

Chapter 3. From Viète to Descartes 63
François Viète 63
Simon Stevin 68
Pierre de Fermat 69
René Descartes 72

Chapter 4. The Predecessors of Galois 76
Waring 76
Vandermonde 77
Lagrange 79
Malfatti 81
Ruffini 83
Cauchy 85
Abel 85

Chapter 5. Carl Friedrich Gauss 89
The Cyclotomic Equation 89
The "Fundamental Theorem" 94
The First Proof 95
The Second Proof 97
The Third Proof 99

Chapter 6. Evariste Galois 103
The Work of Galois 103
The Duel 104
The Memoir of 1831 105
Galois Fields 109
The Publication of Galois' Papers 112
Hermite, Puiseux, and Serret 112
Enrico Betti 114
The Second Posthumous Memoir of Galois 115

Chapter 7. Camille Jordan 117
Jordan's Traité 117
On Groups of Motions 118

On Congruences . . . 121
Transitive and Primitive Groups of Substitutions . . . 121
Series of Composition . . . 121
Linear Substitutions . . . 122
Jordan's Presentation of Galois Theory . . . 124
Geometrical Applications . . . 125
The 28 Double Tangents of a Plane Quartic . . . 128
Application of Galois Theory to Transcendental Functions . . . 131
On Solvable Groups . . . 133

Part Two. Groups

Chapter 8. Early Group Theory . . . 137
Part A. Groups of Substitutions . . . 137
Early Theorems Concerning Subgroups of S_n . . . 137
Mathieu . . . 139
Sylow . . . 139
Part B. Groups of Transformations . . . 140
Non-Euclidean Geometry . . . 141
Felix Klein and Sophus Lie . . . 144
Felix Klein on Finite Groups of Fractional Linear Transformations . 145
Sophus Lie . . . 146
Part C. Abstract Groups . . . 147
Leonhard Euler . . . 147
Carl Friedrich Gauss . . . 148
Ernst Schering . . . 149
Leopold Kronecker . . . 149
Arthur Cayley . . . 150
Walter von Dyck . . . 152
Heinrich Weber . . . 153
Part D. The Structure of Finite Groups . . . 154
Otto Hölder . . . 155
Finite Linear Groups . . . 158

Chapter 9. Lie Groups and Lie Algebras . . . 160
Part A. Lie Groups . . . 160
Lie's Theory . . . 160
Infinitesimal Transformations . . . 162
Three Fundamental Theorems . . . 163
Part B. Lie Algebras . . . 165
Sophus Lie and Friedrich Engel . . . 166
Wilhelm Killing . . . 166
Élie Cartan . . . 168
The Characteristic Roots . . . 168

Semi-Simple Lie Groups 170
Weyl's Group (*S*) 172
Real Simple Lie Algebras 173

Part Three. Algebras

Chapter 10. The Discovery of Algebras 177
Complex Numbers 177
Hamilton's Discovery of Quaternions 179
The Leap into the Fourth Dimension 181
Octonions 183
Product Formulae for Sums of Squares 184
Geometrical Applications of Quaternions 185
The Arithmetic of Quaternions 186
Biquaternions 188
Full Matrix Algebras 189
Group Algebras 190
Grassmann's Calculus of Extensions 191
Clifford Algebras and Rotations in n Dimensions 192
Dirac's Theory of the Spinning Electron 194
Spinors in n Dimensions 195
Chevalley's Generalization 197
Generalized Quaternions 197
Crossed Products 199
Cyclic Algebras 200

Chapter 11. The Structure of Algebras 202
General Notions and Notations 202
Benjamin Peirce 203
Eduard Study 204
Gauss, Weierstrass and Dedekind 204
Georg Scheffers 205
Theodor Molien 206
Élie Cartan 208
Maclagan Wedderburn 210
Emil Artin 210
Emmy Noether and her School 211
Nathan Jacobson 211
Normal Simple Algebras 212
The Structure of Division Algebras 213
1. Division Algebras over $\mathbb{R}$ 214
2. Finite Skew Fields 214
3. Normal Simple Algebras over a P-adic Field 214
4. Division Algebras over an Algebraic Number Field 215

Chapter 12. Group Characters 218
Part A. Characters of Abelian Groups 218
Genera and Characters of Quadratic Forms 218
Duality in Abelian Groups 221
Part B. Characters of Finite Groups 223
Dedekind's Introduction of the Group Determinant 223
Frobenius on the Group Determinant 225
Frobenius on Commuting Matrices 229
The Letter of April 17, 1896 231
The Letter of April 26 233
Frobenius' Paper "Über Gruppencharaktere" 234
The Proof of $e=f$ and the Factorization of the Group Determinant . 236

Chapter 13. Representations of Finite Groups and Algebras 237
Heinrich Maschke 238
Issai Schur 239
Representations of the Symmetric Group 240
The Representation of Groups by Projective Transformations 243
Emmy Noether 244
1. Group-Theoretical Foundations 244
2. Non-Commutative Ideal Theory 245
3. Modules and Representations 246
4. Representations of Groups and Algebras 248

Chapter 14. Representations of Lie Groups and Lie Algebras 252
Cartan's Theory 252
The Global Method 253
The Infinitesimal Method 254
Hermann Weyl 257
John von Neumann 261

Index 265