

# Contents

*Preface* v

## *Introduction. Basic Concepts of Set Theory*

- 0.1. Propositions, propositional functions, sets, 1
- 0.2. Ordered pairs, Cartesian products, 4
- 0.3. Binary relations, 5
- 0.4. Functions, 7
- 0.5. Operations, 9
- 0.6. Identity functions, inverse functions, 10
- 0.7. Similarity of two sets, cardinality, 10
- 0.8. Finite sets, infinite sets, 11
- 0.9. Ordered sets, 11

## *1 The Real Number System*

Introduction, 14

- 1.1. Definition of the real number system, 18
- 1.2. The integers, 21
- 1.3. The rational numbers, 25
- 1.4. Other properties of the real number system, 28
- 1.5. The distance between two real numbers, 29
- 1.6. Limit points and closed and open sets, 30
- 1.7. Sequences, 33
- 1.8. An extension of terminology, 37
- 1.9. Monotonic sequences, 37
- 1.10. Combinations of sequences, 39
- 1.11. Connectedness, 43

## *2 Functions, Mappings, and Graphs*

Introduction, 48

- 2.1. Continuity, 49
- 2.2. Attainment of LUB, GLB, and intermediate values, 53
- 2.3. Graphs, 54

- 2.4. Limits of functions, 57
- 2.5. Roots, 64
- 2.6. Irrational numbers, 65
- 2.7. Limits of certain radical functions, 65
- 2.8. An extension of limit terminology, 67
- 2.9. Uniform continuity, 70
- 2.10. Linear functions, 75

### 3 *The Derivative and the Integral*

- 3.1. The derivative, 77
- 3.2. The Lipschitz property, 86
- 3.3. Maxima and minima, 87
- 3.4. The law of the mean, 93
- 3.5. Differentiability and continuity, 96
- 3.6. Functions of bounded variation, 98
- 3.7. The uniform Lipschitz property, 99
- 3.8. A primitive of a function, 100
- 3.9. The integral, 100

### 4 *Introduction to the Applications of the Integral*

- Introduction, 122
- 4.1. Areas, 123
- 4.2. Centroids of plane regions, 127
- 4.3. Volumes, 134
- 4.4. Disk elements, 135
- 4.5. Shell elements, 137
- 4.6. Sectional elements, 140
- 4.7. Centroids of regions of revolution, 144
- 4.8. Work, 149
- 4.9. Hydrostatic force, 152

### 5 *The Derivative. Introduction to the Applications of the Derivative*

- 5.1. The derivative of the product of two functions, 155
- 5.2. The derivative of the quotient of two functions, 156
- 5.3. The chain rule, 156
- 5.4. The derivative of a power of a function, 157
- 5.5. Implicit functions, 160
- 5.6. Continuity of  $F(x, y)$  at  $(x_0, y_0)$ , 161
- 5.7. The partial derivative, 161
- 5.8. The derivative of an implicit function, 162
- 5.9. Higher derivatives, 162
- 5.10. Taylor's expansion, 163
- 5.11. Graph tracing, 168
- 5.12. Critical points and points of inflection, 169
- 5.13. Symmetry, 169
- 5.14. Graphs of the rational functions, 172
- 5.15. Loci of equations of the form  $y^2 = f(x)$ , 177
- 5.16. Maxima and minima, 182
- 5.17. Time rates, 188
- 5.18. The differential, 193
- 5.19. The differential as a function of two variables, 195
- 5.20. Approximations, 197

## 6 *Plane Analytic Geometry*

- Introduction, 202
- 6.1. The distance between two points, 204
- 6.2. The coordinates of points of division, 204
- 6.3. Loci, 207
- 6.4. The straight line, 212
- 6.5. The slope, 215
- 6.6. The distance between a point and a line, 216
- 6.7. Families of straight lines, 220
- 6.8. Tangents and normals, 225
- 6.9. Symmetry, 228
- 6.10. The circle, 229
- 6.11. Pencils of circles, 231
- 6.12. The conics, 235
- 6.13. The parabola, 238
- 6.14. Geometric properties of the parabola, 241
- 6.15. The ellipse, 244
- 6.16. Geometric properties of the ellipse, 247
- 6.17. The hyperbola, 252
- 6.18. Geometric properties of the hyperbola, 254
- 6.19. Coordinate transformations, 261
- 6.20. Translation of axes, 262
- 6.21. Rotation of axes, 263
- 6.22. Coordinate systems, 265
- 6.23. Curve tracing, 268

## 7 *Elementary Functions*

- 7.1. The logarithmic function  $\ln x$ , 273
- 7.2. An alternative presentation of the logarithmic function, 277
- 7.3. The exponential function  $e^x$ , 278
- 7.4. The exponential function  $a^x$ ,  $a > 0$ , 280
- 7.5. The logarithmic function  $\log_b x$ , where  $b > 0$ , 281
- 7.6. The power function  $x^p$ , where  $x > 0$  and  $p$  is any real number, 282
- 7.7. Logarithmic differentiation, 283
- 7.8. Roots, 284
- 7.9. The trigonometric functions, 285
- 7.10. Other inverse trigonometric functions, 295
- 7.11. The hyperbolic functions, 299
- 7.12. Inverse hyperbolic functions, 302
- 7.13. Parametric equations, 306
- 7.14. Geometric representation of the  $x$  in the trigonometric functions, 310

## 8 *Applications of the Derivative*

- 8.1. Tangents and normals, 312
- 8.2. Maxima and minima, 316
- 8.3. Time rates, 320
- 8.4. Differential equations, 322
- 8.5. Application of the differential equation  $dy/dx = ky$ , 322
- 8.6. Differential equations of the form  $d^n y/dx^n = f(x)$ , 323
- 8.7. Approximate formulas, 325
- 8.8. Evaluation of certain indeterminate forms, 328
- 8.9. Linear motion, 338
- 8.10. Solution of equations in one unknown, 342

## 9 *The Search for Primitives. Applications of the Integral*

- 9.1. The systematic search for primitives, 348
- 9.2. Algebraic substitution, 355
- 9.3. Trigonometric substitution, 356
- 9.4. Integration by parts, 362
- 9.5. Decomposition into partial fractions, 365
- 9.6. Some special types of integrals, 371
- 9.7. Improper integrals of the first kind, 374
- 9.8. Improper integrals of the second kind, 382

## 10 *Further Applications of the Integral*

- 10.1. The length of a curve, 388
- 10.2. Curves, 389
- 10.3. The identification of  $\theta$  in  $\sin \theta$ ,  $\cos \theta$ , and  $\tan \theta$  with the angle  $\theta$ , 397
- 10.4. The centroid of a curve, 398
- 10.5. The moment of inertia of a curve, 401
- 10.6. The area of a surface of revolution, 404
- 10.7. The curvature of a curve at a given point, 407
- 10.8. Applications of the integral in the polar coordinate system, 410
- 10.9. Approximate integration. Simpson's method, 417

## 11 *Infinite Series of Constants*

Introduction, 424

- 11.1. The geometric series, 425
- 11.2. The Cauchy condition for convergence, 426
- 11.3. Tests for convergence and divergence, 428
- 11.4. Operations with series, 442
- 11.5. Approximations to the sum of a convergent series, 445
- 11.6. The decimal representation of the real numbers, 451
- 11.7. The uniqueness of the real number system in the sense of isomorphism, 453

## 12 *Infinite series of Functions. Power Series*

- 12.1. Convergence, 457
- 12.2. Continuity, differentiability, and integrability of the sum of a series, 460
- 12.3. The domain of convergence, 464
- 12.4. The radius of convergence, 466
- 12.5. Properties of power series, 470
- 12.6. Representation of functions by power series, 471
- 12.7. The differential equation  $\frac{dy}{dx} = ky$ ,  $k$  a constant, 480

## 13 *Linear Algebra*

- 13.1. The concept of group, 484
- 13.2. The concept of vector space, 485
- 13.3. Bases, 487
- 13.4. Linear functions, 489
- 13.5. Inner products, 490
- 13.6. The angle between two nonzero vectors, 491
- 13.7. Isomorphism, 495
- 13.8. The vector space  $S_n(F)$ , 496
- 13.9. Linear transformations from  $S^n$  to  $S^m$ . Matrices, 497
- 13.10. Square matrices, 499
- 13.11. Operations with matrices, 500
- 13.12. Determinants, 501

- 13.13. Expansion of determinants, 506
- 13.14. Solution of system of linear equations by determinants, 510
- 13.15. Square matrices, 516
- 13.16. Transformations, 517
- 13.17. Quadratic forms, 524

## 14 The Euclidean Vector Spaces

- Introduction, 527
- 14.1. Functions from  $E^1$  to  $E^n$ , 528
- 14.2. Protovectors, 530
- 14.3. Directed segments, 532
- 14.4. Vectors, 535
- 14.5. Angle between two vectors, 537
- 14.6. Applications of vectors in elementary geometry, 538
- 14.7. An isomorphism, 541
- 14.8. The scalar product of two vectors, 542
- 14.9. A basis for  $E_n$ , 544
- 14.10. The vector space  $E_2$ , 548
- 14.11. The vector space  $E_3$ , 552
- 14.12. The coordinate components of a vector, 553
- 14.13. Vector functions of a real variable, 556
- 14.14. Planar motion, 558
- 14.15. The vector product of two vectors, 567
- 14.16. The scalar triple product, 571
- 14.17. The vector triple product, 572
- 14.18. Vector derivative formulas, 573

## 15 Analytic Geometry of $E^3$

- Introduction, 575
- 15.1. Rays. Straight lines, 576
- 15.2. Direction numbers, 577
- 15.3. Parallelism and orthogonality of lines, 578
- 15.4. The plane, 580
- 15.5. Orthogonality between a line and plane, 582
- 15.6. Parallelism and orthogonality of planes, 587
- 15.7. The normal form of the equation of a plane, 590
- 15.8. Surfaces, 594
- 15.9. The sphere, 595
- 15.10. The ellipsoid, 596
- 15.11. The hyperboloid of one sheet, 596
- 15.12. The hyperboloid of two sheets, 597
- 15.13. The elliptic paraboloid, 598
- 15.14. The hyperbolic hyperboloid, 599
- 15.15. Surfaces of revolution, 600
- 15.16. Cylinders, 601
- 15.17. Cones, 602
- 15.18. Curves and arcs in  $E^3$ , 604
- 15.19. Tangents to an arc in  $E^3$ , 608
- 15.20. Normal plane to an arc at a given point, 611
- 15.21. Length of arcs in  $E^3$ , 611
- 15.22. Motion in  $E^3$  of a particle, 616
- 15.23. Curvature, principal normal, and binormal, 616
- 15.24. The acceleration vector, 617
- 15.25. The cylindrical coordinate system, 619
- 15.26. The spherical coordinate system, 620

## 16 *Functions of Two or More Real Variables*

- Introduction, 622
- 16.1. Neighborhoods, 622
- 16.2. Sequences, 625
- 16.3. Continuity, 625
- 16.4. The partial derivative, 627
- 16.5. The law of the mean. First form, 630
- 16.6. The principal part of the increment of a function, 631
- 16.7. The differential, 636
- 16.8. The chain rule, 641
- 16.9. The uniform Lipschitz property, 644
- 16.10. The law of the mean. Symmetric form, 645
- 16.11. Time rates, 648
- 16.12. The partial derivatives of higher order, 649
- 16.13. An application, 651
- 16.14. The higher differentials, 654
- 16.15. Taylor's expansion, 656
- 16.16. The directional derivative, 660
- 16.17. The gradient, 664
- 16.18. The directional derivative of a function of three variables, 666
- 16.19. The gradient of a function of three variables, 668
- 16.20. Tangent planes and normal lines, 670
- 16.21. Maxima and minima, 673
- 16.22. The general case, 681
- 16.23. Implicit functions, 683
- 16.24. Implicit functions defined by several equations, 689
- 16.25. Transformations or mappings, 699
- 16.26. Products of transformations, 702
- 16.27. Functional dependence, 704
- 16.28. Constrained maxima and minima, 712

## 17 *Multiple Integrals*

- Introduction, 720
- 17.1. Intervals, 720
- 17.2. Limit points, 721
- 17.3. Continuity, 725
- 17.4. Areas and volumes, 728
- 17.5. Multiple integrals, 736
- 17.6. Upper sums and lower sums, 738
- 17.7. Multiple integrals on arbitrary closed, bounded, and connected sets, 744
- 17.8. Iterated integrals, 752
- 17.9. Changing the order of integration, 755
- 17.10. Multiple integrals as iterated integrals, 758
- 17.11. Change of variables, 766
- 17.12. Continuously differentiable transformations, 767
- 17.13. Differentiation under the integral sign, 779
- 17.14. Improper multiple integrals, 785

## 18 *Applications of the Multiple Integral*

- 18.1. Sets in  $E^2$ . Areas, 792
- 18.2. Centroids, 798
- 18.3. Moments of inertia, 803
- 18.4. Moments of inertia with respect to a point, 808

- 18.5. Planar sets with mass, 809
- 18.6. Sets in  $E^3$ . Volumes, 814
- 18.7. Centroids, 825
- 18.8. Moments of inertia, 833
- 18.9. Moments of inertia with respect to a line, 840
- 18.10. Sets with mass, 842
- 18.11. Area of a surface, 844
- 18.12. Improper integrals, 865
- 18.13. Area of a composite surface, 867
- 18.14. Surface area in cylindrical coordinates, 871
- 18.15. Surfaces of revolution, 872

*Appendix A*, 875

*Appendix B*, 877

*Answers to Exercises, Set B*, 882

*Subject Index*, 917