Contents

Preface vii

1 Introduction 1

2 Clifford Algebras and Spinor Operators 5
P. Lounesto
2.1 A History of Clifford Algebras 5
2.2 Teaching Clifford algebras 6
2.2.1 The Clifford Product of Vectors. 7
2.2.2 The Exterior Product. 8
2.2.3 Components of a Vector in Given Directions. 9
2.2.4 Perpendicular Projections and Reflections. 10
2.2.5 Matrix Representation of $C\ell_2$. 11
2.2.6 Exercises 12
2.2.7 Answers 13
2.3 Operator Approach to Spinors 13
2.3.1 Relation to the Convention of Bjorken-Drell. 14
2.3.2 Bilinear Covariants. 16
2.3.3 Fierz Identities (Discovered by Pauli and Kofink). 18
2.3.4 Recovering the Spinor from its Bilinear Covariants. 19
2.3.5 Boomerangs and the Reconstruction of Spinors. 20
2.3.6 The Mother of All Real Spinors $\Phi \in C\ell_{1,3}\frac{1}{2}(1+\gamma_0)$. 22
2.3.7 Ideal Spinors $\phi \in C\ell_{1,3}\frac{1}{2}(1-\gamma_{03})$. 23
2.3.8 Spinor Operators $\Psi \in C\ell^+_{1,3}$. 24
2.3.9 Decomposition and Factorization of Boomerangs. 27
2.4 Flags, Poles and Dipoles 28
2.4.1 A Geometric Classification of Spinors by their Bilinear Covariants. 29
2.4.2 Projection Operators in End($C\ell_{1,3}$). 31
2.4.3 Projection Operators for Majorana and Weyl Spinors. 32

3 Introduction to Geometric Algebras 37
G. Sobczyk
3.1 The Unipodal Number System 37
3.2 Clifford Algebra Matrix Algebra Connection 38

3.3 Geometric algebra 40

4 Linear Transformations **45**
G. Sobczyk
4.1 Structure of a Linear Operator 45
4.2 Isometries 48
4.3 Minimal Polynomials 50
4.4 Lie Algebras of Bivectors 51

5 Directed Integration **53**
G. Sobczyk
5.1 Simplices and Chains 53
5.2 Integral Definition of $\partial_x F(x)$ on S 58
5.3 Classical Integration Theorems 60
5.4 Residue Theorem 61
5.5 Riemannian Geometry 61

6 Linear Algebra **65**
C. Doran, A. Lasenby, and S. Gull
6.1 Geometric Algebra 65
6.2 Spacetime Algebra 66
6.3 Geometric V's Tensor Algebra 69
6.4 Index-Free Linear Algebra 70
6.5 Multivector Calculus 72
6.6 Adjoints and Inverses 74
6.7 Eigenvectors and Eigenbivectors 75
6.8 Invariants 78
6.9 Linear Functions in Spacetime 79
6.10 Functional Differentiation 79

7 Dynamics **83**
S. Gull, C. Doran, and A. Lasenby
7.1 Rigid Body Dynamics 83
7.1.1 The Rotor Equation 83
7.1.2 Kinetic Energy and the Inertia Tensor 84
7.1.3 The Equations of Motion of a Rigid Body 86
7.1.4 Free Precession of a Symmetric Top 87
7.2 Dynamics of Elastic Media 88
7.2.1 Energy Flow 90
7.2.2 Pre-Stressed Media 90
7.2.3 Linearised elasticity 91
7.2.4 The Elastic Filament 92

8 Electromagnetism **95**
S. Gull, C. Doran, and A. Lasenby
8.1 Electromagnetic Waves 97
8.1.1 Stokes Parameters 97
8.1.2 Reflection by a Conducting Plane 98
8.1.3 Waves in layered media 99
8.2 Diffraction Theory 100
8.2.1 The Boundary-Value Problem in Electrodynamics 100
8.3 The Electromagnetic Field of a Point Charge 102
8.4 Applications 104
8.4.1 Uniformly Moving Charge 104
8.4.2 Accelerated Charge 105
8.4.3 Circular Orbits 108

9 Electron Physics I **111**
S. Gull, C. Doran, and A. Lasenby
9.1 Pauli Spinors 111
9.1.1 Pauli Observables 112
9.1.2 Spinors and Rotations 114
9.2 Dirac Spinors 114
9.2.1 Alternative Representations 116
9.3 The Dirac Equation and Observables 117
9.3.1 Plane-Wave States 118
9.4 Hamiltonian Form 120
9.5 The Non-Relativistic Reduction 121
9.6 Angular Eigenstates and Monogenic Functions 123
9.6.1 The Spherical Monogenics 124
9.7 Application — the Coulomb Problem 126

10 Electron Physics II **129**
S. Gull, C. Doran, and A. Lasenby
10.1 Propagation and Characteristic Surfaces 129
10.2 Spinor Potentials and Propagators 130
10.3 Scattering Theory 131
10.3.1 The Born Approximation and Coulomb Scattering 132
10.4 Plane Waves at Potential Steps 133
10.4.1 Matching Conditions for Travelling Waves 135
10.4.2 Matching onto Evanescent Waves 137
10.5 Spin Precession at a Barrier 139
10.6 Tunnelling of Plane Waves 141
10.7 The Klein Paradox 143

11 STA and the Interpretation of Quantum Mechanics **147**
A. Lasenby, S. Gull, and C. Doran
11.1 Tunnelling Times 147

11.2 Spin Measurements 151
11.2.1 A Relativistic Model of a Spin Measurement 152
11.2.2 Wavepacket Simulations 154
11.3 The Multiparticle STA 158
11.3.1 2-Particle Pauli States and the Quantum Correlator 159
11.3.2 Multiparticle Wave Equations 161
11.3.3 The Pauli Principle 163
11.3.4 8-Dimensional Streamlines and Pauli Exclusion 165

12 Gravity I — Introduction **171**
A. Lasenby, C. Doran, and S. Gull
12.1 Gauge Theories 172
12.2 Gauge Principles and Gravitation 175
12.3 The Gravitational Gauge Fields 177
12.3.1 The Rotation-Gauge Field 179
12.4 Observables and Covariant Derivatives 182

13 Gravity II — Field Equations **185**
C. Doran, A. Lasenby, and S. Gull
13.1 The Gravitational Field Equations 186
13.2 Covariant Forms of the Field Equations 189
13.3 Symmetries and Invariants of $\mathcal{R}(B)$ 190
13.4 The Bianchi Identity 193
13.5 Symmetries and Conservation Laws 194

14 Gravity III — First Applications **197**
A. Lasenby, C. Doran, and S. Gull
14.1 Spherically-Symmetric Static Solutions 197
14.1.1 Point-Particle Trajectories 200
14.1.2 Particle Motion in a Spherically-Symmetric Background . . . 202
14.2 Electromagnetism in a Gravitational Background 206
14.2.1 Application to a Black-Hole Background 208

15 Gravity IV — The 'Intrinsic' Method **211**
C. Doran, A. Lasenby, and S. Gull
15.1 Spherically-Symmetric Systems 212
15.2 Two Applications 217
15.3 Stationary, Axially-Symmetric Systems 219
15.4 The Kerr Solution 221

16 Gravity V — Further Applications **223**
A. Lasenby, C. Doran, and S. Gull
16.1 Collapsing Dust and Black Hole Formation 223
16.2 Cosmology 225
16.2.1 The Dirac Equation in a Cosmological Background 229
16.2.2 Point Charge in a $k > 0$ Cosmology 230

16.3 Cosmic Strings 232

17 The Paravector Model of Spacetime **237**
W. E. Baylis
17.1 A Brief Introduction to the Pauli Algebra 237
17.1.1 Generating $\mathcal{C}\ell_3$ 238
17.1.2 Bivectors as Operators 239
17.1.3 Complex Structure 239
17.1.4 Involutions of $\mathcal{C}\ell_3$ 240
17.2 Inverses and the metric 242
17.3 The Spacetime Manifold 244
17.4 Lorentz Transformations I 244
17.5 Vector Notation and Rotations 246
17.5.1 The Merry-Go-Round 247
17.5.2 Observers, Frames, and Vector Bases 248
17.6 Lorentz Transformations II 248
17.6.1 Proper Velocity 250
17.6.2 Covariant vs. Invariant 251
17.6.3 Spacetime Diagrams 251

18 Eigenspinors in Electrodynamics **253**
W. E. Baylis
18.1 Basic Electrodynamics 253
18.2 Eigenspinors 254
18.3 The Group SL(2,C): Diagrams 255
18.4 Time Evolution of Eigenspinor 257
18.4.1 Thomas Precession 258
18.5 Spinorial Lorentz-Force Equation 259
18.5.1 Solutions 259
18.6 Electromagnetic Waves in Vacuum 261
18.6.1 Maxwell's Equation in a Vacuum 261
18.7 Projectors 263
18.8 Directed Plane Waves 264
18.8.1 Polarization 266
18.9 Motion of Charges in Plane Waves 267

19 Eigenspinors in Quantum Theory **269**
W. E. Baylis
19.1 Introduction 269
19.2 Spin 269
19.2.1 Magic of the Pauli Hamiltonian 270
19.2.2 Classical Spin Distribution 270
19.2.3 Quantum Form 271
19.2.4 Stern-Gerlach Filter 271
19.2.5 Linear Combinations of Spatial Rotations 272

19.3 Covariant Eigenspinors 273
19.3.1 Generalized Unimodularity 273
19.3.2 Eigenspinor of an Elementary Particle 273
19.4 Differential-Operator Form 275
19.4.1 The Electromagnetic Gauge Field 277
19.4.2 Linearity and Superposition 278
19.5 Basic Symmetry Transformations 278
19.6 Relation to Standard Form 279
19.6.1 Weyl Spinors 280
19.6.2 Momentum Eigenstates 281
19.6.3 Standing Waves 281
19.6.4 Zitterbewegung 282
19.7 Hamiltonians 282
19.7.1 Stationary States 282
19.7.2 Landau Levels 283
19.8 Fierz Identities of Bilinear Covariants 283

20 Eigenspinors in Curved Spacetime **285**
W. E. Baylis
20.1 Ideals, Spinors, and Symplectic Spaces 285
20.2 Bispinors 287
20.3 Flagpoles and Flags 288
20.4 Spinor Pairs 289
20.5 Time Evolution 290
20.6 Bispinor Basis of $\mathbb{C}^4$ 291
20.7 Twistors 292
20.8 Relation to $SO_+(1,3)$ 293
20.9 Spinors in Curved Spacetime 294
20.10Conclusions 295

21 Spinors: Lorentz Group **297**
J. P. Crawford
21.1 Introduction 297
21.2 Lorentz Group 297
21.2.1 Lorentz Lie Algebra 298
21.2.2 Lorentz Group Representations 299
21.3 Summary 305

22 Spinors: Clifford Algebra **307**
J. P. Crawford
22.1 Introduction 307
22.2 Clifford Algebra 307
22.2.1 Complex Clifford Algebra 308
22.2.2 Automorphism Group 309
22.2.3 Lorentz Group Redux 310

22.2.4 Poincaré Group . . . 313
22.2.5 Conformal Group . . . 314
22.3 Summary . . . 315

23 General Relativity: An Overview **317**
J. P. Crawford
23.1 Introduction . . . 317
23.2 Tensor Analysis . . . 318
23.2.1 Vectors and Tensors . . . 318
23.2.2 Affine Connection and Covariant Differentiation . . . 318
23.2.3 Torsion . . . 319
23.2.4 Parallel Transport and Curvature . . . 320
23.2.5 Bianchi Identities . . . 321
23.2.6 Metric . . . 321
23.2.7 Contracted Bianchi Identities . . . 323
23.3 General Relativity . . . 323
23.3.1 The Principle of Equivalence and the Einstein Equation . . . 323
23.3.2 Gravitational Action . . . 324

24 Spinors in General Relativity **329**
J. P. Crawford
24.1 Introduction . . . 329
24.2 Local Lorentz Invariance . . . 330
24.2.1 Vierbeins . . . 330
24.2.2 Local Lorentz Invariance . . . 331
24.2.3 Covariant Derivative and Spin Connection . . . 331
24.2.4 Lorentz Field Strength Tensor . . . 333
24.2.5 Action and Field Equations . . . 333
24.3 Spinors in Genral Relativity . . . 334
24.3.1 Dirac Equation & The Clifford Algebra . . . 334
24.3.2 Lagrangian and Field Equations . . . 336
24.4 Summary and Conclusions . . . 338

25 Hypergravity I **341**
J. P. Crawford
25.1 Introduction . . . 341
25.2 Automorphism Invariance . . . 343
25.2.1 Automorphism Group . . . 343
25.2.2 Covariant Derivative and Drehbeins . . . 344
25.2.3 Curvatures and Field Strength Tensors . . . 347
25.2.4 Bianchi Identities . . . 349
25.3 Discussion . . . 350

26 Hypergravity II **353**
J. P. Crawford
26.1 Introduction 353
26.2 Lagrangian 354
26.2.1 Gauge Field Terms 354
26.2.2 Spinor Field Terms 354
26.2.3 Drehbein Terms 355
26.3 Field Equations 355
26.3.1 Drehbein Field Equations 356
26.3.2 Spinor Field Equations 357
26.3.3 Gauge Field Equations 357
26.3.4 Gravitational Field Equations 359
26.4 Einstein Gravity Recovered 361
26.5 Discussion 362

27 Properties of Clifford Algebras for Fundamental Particles **365**
J. S. R. Chisholm and R. S. Farwell
27.1 Building Blocks of a Gauge Model 365
27.1.1 Introduction 365
27.1.2 The Elements of the Algebra 366
27.1.3 Operations within the Algebra 370
27.2 Spinors in the Clifford Algebra $\mathcal{C}\ell_{p,q}$ 374
27.2.1 Minimal Left Ideals of $\mathcal{C}\ell_{p,q}$ 374
27.2.2 Spinors in $\mathcal{C}\ell_{p,q}$ 376
27.2.3 Bar-Conjugate Spinors in $\mathcal{C}\ell_{p,q}$ 377
27.2.4 The Spinor Norm in $\mathcal{C}\ell_{p,q}$ 379
27.2.5 Left- and Right-Handed Spinors in $\mathcal{C}\ell_{p,q}$ 380
27.3 Selecting a Higher-Dimensional Algebra for a Gauge Model 382
27.3.1 The Principles of the Model 382
27.3.2 A Model Based on $C\ell_{1,6}$ 384
27.3.3 A Gauge Model in $C\ell_{1,6}$ 385

28 The Extended Grassmann Algebra of $\mathbf{R}^3$ **389**
B. Jancewicz
28.1 Introduction 389
28.2 Multivectors and pseudo-multivectors 391
28.3 Forms and pseudoforms 397
28.4 Linear spaces 403
28.5 Various products 408
28.6 Physical quantities 411
28.7 Quadratic spaces 414
28.8 Conclusion 419

29 Geometric Algebra: Applications in Engineering **423**
J. Lasenby
29.1 Applications in Computer Vision 423
29.1.1 Projective Space and Projective Transformations 423
29.1.2 Geometric Invariance in Computer Vision 426
29.1.3 Motion and Structure from Motion 429
29.2 Applications in Robotics/Mechanisms 434
29.2.1 Screw Transformations 435
29.2.2 A simple robot arm 436
29.2.3 Dual Quaternions 438
29.3 Further Applications 440

30 Projective Quadrics, Poles, Polars, and Legendre Transformations **441**
R. C. Pappas
30.1 Introduction 441
30.2 The Legendre Transformation (1) 442
30.3 Projective Spaces and Geometric Algebra 443
30.4 Quadrics, poles, and polars 443
30.5 The Legendre Transformation (2) 446
30.6 Comments 447

31 Spacetime Algebra and Line Geometry **449**
J. G. Maks
31.1 Introduction 449
31.2 Projective geometry 450
31.3 The null polarity belonging to $R^{2,3}$ 452
31.4 Line geometry in spacetime algebra 454

32 Generalizations of Clifford Algebra **459**
C. C. Carbno
32.1 Generalizations of Clifford Algebra 459
32.1.1 Introduction 459
32.2 Dimensions of Zero Extent 460
32.3 Bosonic Vectors 461
32.4 Higher Cycling Dimensions 461
32.5 Conclusion 462

33 Clifford Algebra Computations with Maple **463**
R. Abłamowicz
33.1 Introduction 463
33.2 Basic chores. 466
33.2.1 'cliterms' 466
33.2.2 'clisort' 467
33.2.3 'clicollect' 467
33.2.4 'reorder' 467
33.2.5 'scalarpart' 468

33.2.6 'vectorpart' . . . 468
33.3 Ring operations in Clifford algebra and computation of a symbolic inverse. . . . 469
33.3.1 'LC': left contraction by a vector and 'wedge' multiplication. . 469
33.3.2 Clifford multiplication 'cmul' or '&c'. . . . 471
33.3.3 'cinv': symbolic inverse of a multivector. . . . 473
33.4 Clifford algebra automorphisms: grade involution, reversion and conjugation. . . . 475
33.4.1 Grade involution and reversion. . . . 475
33.4.2 Clifford conjugation and complex conjugation. . . . 477
33.5 Matrix representations. . . . 478
33.5.1 Left regular representations. . . . 478
33.5.2 Spinor representations in left minimal ideals. . . . 480
33.6 Clifford and exterior exponentiations. . . . 484
33.6.1 'cexp': Clifford exponentiation. . . . 484
33.6.2 'wexp': exterior exponentiation. . . . 486
33.7 Quaternions and three dimensional rotations. . . . 488
33.7.1 Quaternion type, conjugation, norm and inverse. . . . 488
33.7.2 'rot3d': rotations in three dimensions. . . . 489
33.8 Octonions: type, conjugation, norm and inverse. . . . 492
33.9 Working with homomorphisms of algebras. . . . 494

Index **503**