

# CONTENTS

|  |     |
|--|-----|
| Introduction   | xi  |
| 1 The basic functions and definition schemes                       | 1   |
| 1. The basic functions   | 1   |
| 2. The elementary functions  | 3   |
| 3. Primitive recursion and some generalizations                    | 10  |
| 4. Elimination of parameters                                       | 17  |
| 5. Examples 1  | 25  |
| 2 Hierarchy theory   | 29  |
| 1. Historical introduction   | 29  |
| 2. Grzegorczyk's original hierarchy                                | 31  |
| 3. The basic framework for hierarchy theory                        | 37  |
| 4. The hierarchy $\{\mathcal{F}_A^\alpha \mid \alpha \in \Delta\}$ | 44  |
| 5. Hierarchies based on enumeration                                | 48  |
| 6. Examples 2  | 55  |
| 3 Ordinal and nested recursion                                     | 57  |
| 1. Ordinal recursion   | 57  |
| 2. The ordering $\prec^*$  | 60  |
| 3. Nested recursion  | 62  |
| 4. Examples 3  | 75  |
| 4 The extended Grzegorczyk hierarchy                               | 77  |
| 1. The Grzegorczyk and Hardy hierarchies                           | 78  |
| 2. A system of notation for ordinals less than $\epsilon_0$        | 85  |
| 3. The relationship between the main hierarchies                   | 89  |
| 4. The class $\mathcal{E}^*$                                       | 96  |
| 5. Examples 4  | 108 |
| 5 The small classes  | 110 |
| 1. Review of the primitive recursive hierarchies                   | 111 |
| 2. The small 'Grzegorczyk-type' classes                            | 116 |
| 3. The small relational classes                                    | 121 |
| 4. Polynomial time   | 126 |
| 5. Machine versus recursive classes                                | 132 |
| 6. The spectral problem  | 134 |
| 7. Examples 5  | 135 |

## CONTENTS

|    |   |     |
|----|---|-----|
| 6  | $\mathcal{E}^\alpha$ -arithmetic                      | 137 |
| 1. | Codification of $\mathcal{L}^2$ -arithmetic           | 139 |
| 2. | Codification of $\mathcal{E}^\alpha$ -arithmetic      | 148 |
| 3. | Consistency of $\mathcal{E}^\alpha$ -arithmetic       | 149 |
| 4. | The Gödel incompleteness theorems                     | 159 |
| 5. | Examples 6  | 163 |
| 7  | Transfinite induction                                 | 165 |
| 1. | Some generalizations of the basic induction scheme    | 165 |
| 2. | Inductions in $\mathcal{E}^\alpha$ -arithmetic—Part 1 | 168 |
| 3. | Inductions in $\mathcal{E}^\alpha$ -arithmetic—Part 2 | 174 |
| 4. | The ‘Paris–Harrington’ independence results           | 178 |
| 5. | Examples 7  | 181 |
|    | References  | 183 |
|    | Index of notation                                     | 187 |
|    | General index   | 190 |