

Contents

Chapter 1. Introduction	1
1.1. General summary	1
1.2. The ring \mathcal{R}_m of polynomials to be employed throughout	2
1.3. Terminology used throughout (except as modified in Chapter 12)	3
1.4. Principal results	5
1.5. Symbolism for polynomials in \mathcal{R}_m	7
1.6. Miscellaneous observations	8
1.7. Order of presentation	9
Chapter 2. Some Problems of Historical Importance	12
2.1. The older semi-invariants	12
2.2. A challenging problem posed by J. Liouville in 1839	13
2.3. The first construction of a decisive set for any $m \geq 3$	16
2.4. Decisive sets of semi-invariants	18
2.5. Awkward formulations involving the older semi-invariants	19
Chapter 3. Illustrations for Some Results in Chapters 1 and 2	21
3.1. $\{\mathbf{G}_2, \dots, \mathbf{G}_m\}$ and $\{\mathbf{G}_3, \dots, \mathbf{G}_m\}$ are not decisive sets	21
3.2. A simple check on the consistency of (1.20)–(1.27)	22
3.3. A simple check on the consistency of (1.12)–(1.15)	23
3.4. Two types of symbolic sums and their evaluation	24
3.5. Computations when m is a symbol for an integer ≥ 3	27
3.6. A comprehensive check on the consistency of (1.8)–(1.27)	28
Chapter 4. L_n and $I_{n,i}$ as Semi-Invariants of the First Kind	31
Chapter 5. V_n and $J_{n,i}$ as Semi-Invariants of the Second Kind	34
Chapter 6. The Coefficients of Transformed Equations	39
6.1. Alternative formulas for $c_i^{**}(\zeta)$ in (1.5)	39
6.2. The coefficients of a composite transformation	40
6.3. Several examples	44
6.4. Proof of an old observation	45
6.5. Conditions for transformed equations	46
6.6. Formulas for later reference	49
Chapter 7. Formulas That Involve $L_n(z)$ or $I_{n,n}(z)$	50
7.1. The coefficients of (6.8) when $d_1(\zeta) \equiv d_2(\zeta) \equiv 0$	50
7.2. Derivatives for the coefficients of (6.8) when $d_1(\zeta) \equiv d_2(\zeta) \equiv 0$	53
7.3. Identities for the coefficients of (6.8) when $d_1(\zeta) \equiv d_2(\zeta) \equiv 0$	55

Chapter 8. Formulas That Involve $V_n(z)$ or $J_{n,n}(z)$	61
8.1. The coefficients of (6.8) when $d_1(\zeta) \equiv d_2(\zeta) \equiv 0$	61
8.2. Derivatives for the coefficients of (6.8) when $d_1(\zeta) \equiv d_2(\zeta) \equiv 0$	64
8.3. Identities for the coefficients of (6.8) when $d_1(\zeta) \equiv d_2(\zeta) \equiv 0$	66
Chapter 9. Verification of $I_{n,n} \equiv J_{n,n}$ and Various Observations	71
9.1. Proof for the first part of the Main Theorem in Chapter 1	71
9.2. Global sets	73
9.3. A fourth type of invariant: an absolute invariant	74
9.4. Laguerre-Forsyth canonical forms	75
Chapter 10. The Local Constructions of Earlier Research	77
10.1. Standard techniques	77
10.2. An improved computational procedure	78
10.3. Hindrances to earlier research	79
Chapter 11. Relations for G_i , H_i , and L_i That Yield Equivalent Formulas for Basic Relative Invariants	81
11.1. The identity $H_i \equiv G_i - D_{i,m-i}$	81
11.2. Formulas for L_0, \dots, L_m in terms of G_0, \dots, G_m	83
11.3. Formulas for L_3, \dots, L_m in terms of H_3, \dots, H_m	87
11.4. Formulas for G_0, \dots, G_m in terms of L_0, \dots, L_m	90
11.5. Formulas for H_3, \dots, H_m in terms of L_3, \dots, L_m	93
Chapter 12. Real-Valued Functions of a Real Variable	94
12.1. A context suitable for the required evaluations	94
12.2. $\mathcal{I}_{m,3}, \dots, \mathcal{I}_{m,m}$ are relative invariants of global character	95
12.3. Decisive sets for Context 12.1	96
12.4. Global sets	100
Chapter 13. A Constructive Method for Imposing Conditions on Laguerre-Forsyth Canonical Forms	104
13.1. Conditions imposed directly on the coefficients of (1.1)	104
13.2. Illustrations of the technique	106
Chapter 14. Additional Formulas for $K_{i,j}$, $U_{i,j}$, $A_{i,j}$, $D_{i,j}$, ...	108
14.1. Alternative formulas for $K_{i,j}$ in (1.12)–(1.14)	108
14.2. Alternative formulas for $U_{i,j}$ in (1.21)–(1.23)	109
14.3. Alternative formulas for $A_{i,j}$ in (2.19)–(2.21)	110
14.4. Alternative formulas for $D_{i,j}$ in (2.42)–(2.44)	110
14.5. Alternative formulas for $\beta_{i,j}(z)$ in (6.2)–(6.3)	111
14.6. Alternative formulas for $\phi_{i,j}(z)$ in (6.10)–(6.11)	112
14.7. Polynomials $Q_{i,j}$ and R_i for use in Section 15.2	112
Chapter 15. Three Canonical Forms Are Now Available	114
15.1. Reduction of (1.1) to a Halphen canonical form	114
15.2. A neglected canonical form for any (1.1) having $m \geq 2$	116
15.3. Semi-invariants R_2, \dots, R_m analogous to G_2, \dots, G_m	119
15.4. $\{R_2, \dots, R_m\}$ and $\{R_3, \dots, R_m\}$ are not decisive sets	121
15.5. Semi-invariants S_3, \dots, S_m analogous to H_3, \dots, H_m	121
15.6. The identity $S_i \equiv R_i - T_{i,m-i}$	122

15.7. Solving (14.28) for w_i in terms of R_0, \dots, R_m	125
Chapter 16. Interesting Problems that Require Further Study	130
16.1. Constructive characterizations for Fano's research	130
16.2. Generalizations based on Lié groups	132
Appendix A. Results Needed for Self-Containment	133
A.1. The coefficients $c_i^*(z)$ for (1.3)	134
A.2. The coefficients $c_i^{**}(\zeta)$ for (1.5)	135
A.3. Semi-invariants of the second kind given by $S^{(1)} + kb_1 S$	138
A.4. Non-solutions for non-zero equations	139
A.5. Semi-invariants of the second kind are isobaric	140
A.6. Supplementary observations about invariants	142
A.7. Identities relating the coefficients of (1.3) or (1.5) to (1.1)	146
A.8. Pencil-and-paper computation for $I_{3,3}$	148
A.9. Machine computations for the coefficients $c_i(z)$ of (15.32)	149
Appendix B. Related Studies for a Class of Nonlinear Equations	151
B.1. P. Appell's influence on our solution of J. Liouville's problem	151
B.2. Deduction of (B.16) from [12, page 139, Theorem 4.1]	154
Appendix C. Polynomials That Are Linear in a Key Variable	156
C.1. Some polynomials that are not relative invariants	156
C.2. The uniqueness of basic relative invariants and a proof for the second part of the Main Theorem in Chapter 1	163
Appendix D. Rational Semi-Invariants and Relative Invariants	165
D.1. Introduction	165
D.2. Definitions of rational semi-invariants and relative invariants	166
D.3. The integer s in Definition D.2	166
D.4. A context for the remainder of this appendix	169
D.5. A technical construction needed for Section D.6	172
D.6. Rational semi-invariants of the first kind	176
D.7. A technical construction needed for Section D.8	180
D.8. Rational semi-invariants of the second kind	185
D.9. The structure of rational relative invariants	188
D.10. The structure of absolute invariants	188
D.11. Substitutions into fractions of \mathcal{Q}_m	189
Appendix E. Generating Additional Relative Invariants	192
E.1. Two constructions due to G.-H. Halphen and A. R. Forsyth	192
E.2. Examples for the constructions	194
E.3. Historical observations	195
E.4. A challenging problem for further research	195
Bibliography	197
Index	200