

Contents

Preface to the English Edition	xiii
Preface	xvii
Chapter 1. Cartesian Spaces and Euclidean Geometry	1
1.1. Coordinates. Space-time	1
1.1.1. Cartesian coordinates	1
1.1.2. Change of coordinates	2
1.2. Euclidean geometry and linear algebra	6
1.2.1. Vector spaces and scalar products	6
1.2.2. The length of a curve	10
1.3. Affine transformations	12
1.3.1. Matrix formalism. Orientation	12
1.3.2. Affine group	14
1.3.3. Motions of Euclidean spaces	19
1.4. Curves in Euclidean space	24
1.4.1. The natural parameter and curvature	24
1.4.2. Curves on the plane	26
1.4.3. Curvature and torsion of curves in \mathbb{R}^3	28
Exercises to Chapter 1	32
Chapter 2. Symplectic and Pseudo-Euclidean Spaces	35
2.1. Geometric structures in linear spaces	35
2.1.1. Pseudo-Euclidean and symplectic spaces	35
2.1.2. Symplectic transformations	39
2.2. The Minkowski space	43

2.2.1.	The event space of the special relativity theory	43
2.2.2.	The Poincaré group	46
2.2.3.	Lorentz transformations	48
Exercises to Chapter 2		50
Chapter 3. Geometry of Two-Dimensional Manifolds		53
3.1.	Surfaces in three-dimensional space	53
3.1.1.	Regular surfaces	53
3.1.2.	Local coordinates	56
3.1.3.	Tangent space	58
3.1.4.	Surfaces as two-dimensional manifolds	59
3.2.	Riemannian metric on a surface	62
3.2.1.	The length of a curve on a surface	62
3.2.2.	Surface area	65
3.3.	Curvature of a surface	67
3.3.1.	On the notion of the surface curvature	67
3.3.2.	Curvature of lines on a surface	68
3.3.3.	Eigenvalues of a pair of scalar products	70
3.3.4.	Principal curvatures and the Gaussian curvature	73
3.4.	Basic equations of the theory of surfaces	75
3.4.1.	Derivational equations as the “zero curvature” condition. Gauge fields	75
3.4.2.	The Codazzi and sine-Gordon equations	78
3.4.3.	The Gauss theorem	80
Exercises to Chapter 3		81
Chapter 4. Complex Analysis in the Theory of Surfaces		85
4.1.	Complex spaces and analytic functions	85
4.1.1.	Complex vector spaces	85
4.1.2.	The Hermitian scalar product	87
4.1.3.	Unitary and linear-fractional transformations	88
4.1.4.	Holomorphic functions and the Cauchy–Riemann equations	90
4.1.5.	Complex-analytic coordinate changes	92
4.2.	Geometry of the sphere	94
4.2.1.	The metric of the sphere	94
4.2.2.	The group of motions of a sphere	96
4.3.	Geometry of the pseudosphere	100
4.3.1.	Space-like surfaces in pseudo-Euclidean spaces	100
4.3.2.	The metric and the group of motions of the pseudosphere	102

4.3.3. Models of hyperbolic geometry	104
4.3.4. Hilbert's theorem on impossibility of imbedding the pseudosphere into \mathbb{R}^3	106
4.4. The theory of surfaces in terms of a conformal parameter	107
4.4.1. Existence of a conformal parameter	107
4.4.2. The basic equations in terms of a conformal parameter	110
4.4.3. Hopf differential and its applications	112
4.4.4. Surfaces of constant Gaussian curvature. The Liouville equation	113
4.4.5. Surfaces of constant mean curvature. The sinh-Gordon equation	115
4.5. Minimal surfaces	117
4.5.1. The Weierstrass–Enneper formulas for minimal surfaces	117
4.5.2. Examples of minimal surfaces	120
Exercises to Chapter 4	122
Chapter 5. Smooth Manifolds	125
5.1. Smooth manifolds	125
5.1.1. Topological and metric spaces	125
5.1.2. On the notion of smooth manifold	129
5.1.3. Smooth mappings and tangent spaces	133
5.1.4. Multidimensional surfaces in \mathbb{R}^n . Manifolds with boundary	137
5.1.5. Partition of unity. Manifolds as multidimensional surfaces in Euclidean spaces	141
5.1.6. Discrete actions and quotient manifolds	143
5.1.7. Complex manifolds	145
5.2. Groups of transformations as manifolds	156
5.2.1. Groups of motions as multidimensional surfaces	156
5.2.2. Complex surfaces and subgroups of $GL(n, \mathbb{C})$	163
5.2.3. Groups of affine transformations and the Heisenberg group	165
5.2.4. Exponential mapping	166
5.3. Quaternions and groups of motions	170
5.3.1. Algebra of quaternions	170
5.3.2. The groups $SO(3)$ and $SO(4)$	171
5.3.3. Quaternion-linear transformations	173
Exercises to Chapter 5	175
Chapter 6. Groups of Motions	177
6.1. Lie groups and algebras	177

6.1.1.	Lie groups	177
6.1.2.	Lie algebras	179
6.1.3.	Main matrix groups and Lie algebras	187
6.1.4.	Invariant metrics on Lie groups	193
6.1.5.	Homogeneous spaces	197
6.1.6.	Complex Lie groups	204
6.1.7.	Classification of Lie algebras	206
6.1.8.	Two-dimensional and three-dimensional Lie algebras	209
6.1.9.	Poisson structures	212
6.1.10.	Graded algebras and Lie superalgebras	217
6.2.	Crystallographic groups and their generalizations	221
6.2.1.	Crystallographic groups in Euclidean spaces	221
6.2.2.	Quasi-crystallographic groups	232
	Exercises to Chapter 6	242
	Chapter 7. Tensor Algebra	245
7.1.	Tensors of rank 1 and 2	245
7.1.1.	Tangent space and tensors of rank 1	245
7.1.2.	Tensors of rank 2	249
7.1.3.	Transformations of tensors of rank at most 2	250
7.2.	Tensors of arbitrary rank	251
7.2.1.	Transformation of components	251
7.2.2.	Algebraic operations on tensors	253
7.2.3.	Differential notation for tensors	256
7.2.4.	Invariant tensors	258
7.2.5.	A mechanical example: strain and stress tensors	259
7.3.	Exterior forms	261
7.3.1.	Symmetrization and alternation	261
7.3.2.	Skew-symmetric tensors of type $(0, k)$	262
7.3.3.	Exterior algebra. Symmetric algebra	264
7.4.	Tensors in the space with scalar product	266
7.4.1.	Raising and lowering indices	266
7.4.2.	Eigenvalues of scalar products	268
7.4.3.	Hodge duality operator	270
7.4.4.	Fermions and bosons. Spaces of symmetric and skew-symmetric tensors as Fock spaces	271
7.5.	Polyvectors and the integral of anticommuting variables	278
7.5.1.	Anticommuting variables and superalgebras	278
7.5.2.	Integral of anticommuting variables	281
	Exercises to Chapter 7	283
	Chapter 8. Tensor Fields in Analysis	285

8.1. Tensors of rank 2 in pseudo-Euclidean space	285
8.1.1. Electromagnetic field	285
8.1.2. Reduction of skew-symmetric tensors to canonical form	287
8.1.3. Symmetric tensors	289
8.2. Behavior of tensors under mappings	291
8.2.1. Action of mappings on tensors with superscripts	291
8.2.2. Restriction of tensors with subscripts	292
8.2.3. The Gauss map	294
8.3. Vector fields	296
8.3.1. Integral curves	296
8.3.2. Lie algebras of vector fields	299
8.3.3. Linear vector fields	301
8.3.4. Exponential function of a vector field	303
8.3.5. Invariant fields on Lie groups	304
8.3.6. The Lie derivative	306
8.3.7. Central extensions of Lie algebras	309
Exercises to Chapter 8	312
Chapter 9. Analysis of Differential Forms	315
9.1. Differential forms	315
9.1.1. Skew-symmetric tensors and their differentiation	315
9.1.2. Exterior differential	318
9.1.3. Maxwell equations	321
9.2. Integration of differential forms	322
9.2.1. Definition of the integral	322
9.2.2. Integral of a form over a manifold	327
9.2.3. Integrals of differential forms in \mathbb{R}^3	329
9.2.4. Stokes theorem	331
9.2.5. The proof of the Stokes theorem for a cube	335
9.2.6. Integration over a superspace	337
9.3. Cohomology	339
9.3.1. De Rham cohomology	339
9.3.2. Homotopy invariance of cohomology	341
9.3.3. Examples of cohomology groups	343
Exercises to Chapter 9	349
Chapter 10. Connections and Curvature	351
10.1. Covariant differentiation	351
10.1.1. Covariant differentiation of vector fields	351
10.1.2. Covariant differentiation of tensors	357
10.1.3. Gauge fields	359

10.1.4. Cartan connections	362
10.1.5. Parallel translation	363
10.1.6. Connections compatible with a metric	365
10.2. Curvature tensor	369
10.2.1. Definition of the curvature tensor	369
10.2.2. Symmetries of the curvature tensor	372
10.2.3. The Riemann tensors in small dimensions, the Ricci tensor, scalar and sectional curvatures	374
10.2.4. Tensor of conformal curvature	377
10.2.5. Tetrad formalism	380
10.2.6. The curvature of invariant metrics of Lie groups	381
10.3. Geodesic lines	383
10.3.1. Geodesic flow	383
10.3.2. Geodesic lines as shortest paths	386
10.3.3. The Gauss–Bonnet formula	389
Exercises to Chapter 10	392
Chapter 11. Conformal and Complex Geometries	397
11.1. Conformal geometry	397
11.1.1. Conformal transformations	397
11.1.2. Liouville’s theorem on conformal mappings	400
11.1.3. Lie algebra of a conformal group	402
11.2. Complex structures on manifolds	404
11.2.1. Complex differential forms	404
11.2.2. Kähler metrics	407
11.2.3. Topology of Kähler manifolds	411
11.2.4. Almost complex structures	414
11.2.5. Abelian tori	417
Exercises to Chapter 11	421
Chapter 12. Morse Theory and Hamiltonian Formalism	423
12.1. Elements of Morse theory	423
12.1.1. Critical points of smooth functions	423
12.1.2. Morse lemma and transversality theorems	427
12.1.3. Degree of a mapping	436
12.1.4. Gradient systems and Morse surgeries	439
12.1.5. Topology of two-dimensional manifolds	448
12.2. One-dimensional problems: Principle of least action	453
12.2.1. Examples of functionals (geometry and mechanics). Variational derivative	453
12.2.2. Equations of motion (examples)	457

12.3.	Groups of symmetries and conservation laws	460
12.3.1.	Conservation laws of energy and momentum	460
12.3.2.	Fields of symmetries	461
12.3.3.	Conservation laws in relativistic mechanics	463
12.3.4.	Conservation laws in classical mechanics	466
12.3.5.	Systems of relativistic particles and scattering	470
12.4.	Hamilton's variational principle	472
12.4.1.	Hamilton's theorem	472
12.4.2.	Lagrangians and time-dependent changes of coordinates	474
12.4.3.	Variational principles of Fermat type	477
	Exercises to Chapter 12	479
Chapter 13. Poisson and Lagrange Manifolds		481
13.1.	Symplectic and Poisson manifolds	481
13.1.1.	g -gradient systems and symplectic manifolds	481
13.1.2.	Examples of phase spaces	484
13.1.3.	Extended phase space	491
13.1.4.	Poisson manifolds and Poisson algebras	492
13.1.5.	Reduction of Poisson algebras	497
13.1.6.	Examples of Poisson algebras	498
13.1.7.	Canonical transformations	504
13.2.	Lagrangian submanifolds and their applications	507
13.2.1.	The Hamilton–Jacobi equation and bundles of trajectories	507
13.2.2.	Representation of canonical transformations	512
13.2.3.	Conical Lagrangian surfaces	514
13.2.4.	The “action-angle” variables	516
13.3.	Local minimality condition	521
13.3.1.	The second-variation formula and the Jacobi operator	521
13.3.2.	Conjugate points	527
	Exercises to Chapter 13	528
Chapter 14. Multidimensional Variational Problems		531
14.1.	Calculus of variations	531
14.1.1.	Introduction. Variational derivatives	531
14.1.2.	Energy-momentum tensor and conservation laws	535
14.2.	Examples of multidimensional variational problems	542
14.2.1.	Minimal surfaces	542
14.2.2.	Electromagnetic field equations	544
14.2.3.	Einstein equations. Hilbert functional	548

14.2.4. Harmonic functions and the Hodge expansion	553
14.2.5. The Dirichlet functional and harmonic mappings	558
14.2.6. Massive scalar and vector fields	563
Exercises to Chapter 14	566
 Chapter 15. Geometric Fields in Physics	 569
15.1. Elements of Einstein's relativity theory	569
15.1.1. Principles of special relativity	569
15.1.2. Gravitation field as a metric	573
15.1.3. The action functional of a gravitational field	576
15.1.4. The Schwarzschild and Kerr metrics	578
15.1.5. Interaction of matter with gravitational field	581
15.1.6. On the concept of mass in general relativity theory	584
15.2. Spinors and the Dirac equation	587
15.2.1. Automorphisms of matrix algebras	587
15.2.2. Spinor representation of the group $\text{SO}(3)$	589
15.2.3. Spinor representation of the group $\text{O}(1, 3)$	591
15.2.4. Dirac equation	594
15.2.5. Clifford algebras	597
15.3. Yang–Mills fields	598
15.3.1. Gauge-invariant Lagrangians	598
15.3.2. Covariant differentiation of spinors	603
15.3.3. Curvature of a connection	605
15.3.4. The Yang–Mills equations	606
15.3.5. Characteristic classes	609
15.3.6. Instantons	612
Exercises to Chapter 15	616
Bibliography	621
Index	625