Contents

Preface xi

Chapter 1. Eigenvalues and the Laplacian of a graph 1

1.1. Introduction 1

1.2. The Laplacian and eigenvalues 2

1.3. Basic facts about the spectrum of a graph 6

1.4. Eigenvalues of weighted graphs 12

1.5. Eigenvalues and random walks 14

Chapter 2. Isoperimetric problems 23

2.1. History 23

2.2. The Cheeger constant of a graph 24

2.3. The edge expansion of a graph 25

2.4. The vertex expansion of a graph 29

2.5. A characterization of the Cheeger constant 32

2.6. Isoperimetric inequalities for cartesian products 35

Chapter 3. Diameters and eigenvalues 41

3.1. The diameter of a graph 41

3.2. Eigenvalues and distances between two subsets 43

3.3. Eigenvalues and distances among many subsets 46

3.4. Eigenvalue upper bounds for manifolds 48

Chapter 4. Paths, flows, and routing 55

4.1. Paths and sets of paths 55
4.2. Flows and Cheeger constants 56
4.3. Eigenvalues and routes with small congestion 58
4.4. Routing in graphs 60
4.5. Comparison theorems 64

Chapter 5. Eigenvalues and quasi-randomness 69
5.1. Quasi-randomness 69
5.2. The discrepancy property 71
5.3. The deviation of a graph 77
5.4. Quasi-random graphs 81

Chapter 6. Expanders and explicit constructions 87
6.1. Probabilistic methods versus explicit constructions 87
6.2. The expanders 88
6.3. Examples of explicit constructions 93
6.4. Applications of expanders in communication networks 98
6.5. Constructions of graphs with small diameter and girth 101
6.6. Weighted Laplacians and the Lovász ϑ function 103

Chapter 7. Eigenvalues of symmetrical graphs 109
7.1. Symmetrical graphs 109
7.2. Cheeger constants of symmetrical graphs 110
7.3. Eigenvalues of symmetrical graphs 112
7.4. Distance transitive graphs 114
7.5. Eigenvalues and group representation theory 117
7.6. The vibrational spectrum of a graph 119

Chapter 8. Eigenvalues of subgraphs with boundary conditions 123
8.1. Neumann eigenvalues and Dirichlet eigenvalues 123

8.2. The Neumann eigenvalues of a subgraph 124
8.3. Neumann eigenvalues and random walks 126
8.4. Dirichlet eigenvalues 128
8.5. A matrix-tree theorem and Dirichlet eigenvalues 129
8.6. Determinants and invariant field theory 131

Chapter 9. Harnack inequalities 135
9.1. Eigenfunctions 135
9.2. Convex subgraphs of homogeneous graphs 136
9.3. A Harnack inequality for homogeneous graphs 138
9.4. Harnack inequalities for Dirichlet eigenvalues 140
9.5. Harnack inequalities for Neumann eigenvalues 142
9.6. Eigenvalues and diameters 144

Chapter 10. Heat kernels 145
10.1. The heat kernel of a graph and its induced subgraphs 145
10.2. Basic facts on heat kernels 146
10.3. An eigenvalue inequality 148
10.4. Heat kernel lower bounds 150
10.5. Matrices with given row and column sums 156
10.6. Random walks and the heat kernel 161

Chapter 11. Sobolev inequalities 163
11.1. The isoperimetric dimension of a graph 163
11.2. An isoperimetric inequality 165
11.3. Sobolev inequalities 168
11.4. Eigenvalue bounds 170
11.5. Generalizations to weighted graphs and subgraphs 175

Chapter 12. Advanced techniques for random walks on graphs 177

12.1. Several approaches for bounding convergence 177

12.2. Logarithmic Sobolev inequalities 180

12.3. A comparison theorem for the log-Sobolev constant 185

12.4. Logarithmic Harnack inequalities 187

12.5. The isoperimetric dimension and the Sobolev inequality 191

Bibliography 195

Index 204