

CONTENTS

Dedication	v
Foreword to the third American edition	viii
Foreword to the second (= revised first) American edition	viii
Foreword to the second Soviet edition	xiv
From the foreword to the first Soviet edition	xv

CHAPTER I BASIC IDEAS FROM TOPOLOGY AND FUNCTIONAL ANALYSIS

§ 1. Linear spaces	1
1. Definition of a linear space (1). 2. Linear dependence and independence of vectors (2). 3. Subspaces (3). 4. Quotient space (4). 5. Linear operators (5). 6. Operator calculus (8). 7. Invariant subspaces (12). 8. Convex sets and Minkowski functionals (12). 9. Theorems on the extension of a linear functional (16).	
§ 2. Topological spaces	21
1. Definition of a topological space (21). 2. Interior of a set; neighborhoods (22). 3. Closed sets; closure of a set (23). 4. Subspaces (24). 5. Mappings of topological spaces (24). 6. Compact sets (26). 7. Hausdorff spaces (27). 8. Normal spaces (28). 9. Locally compact spaces (30). 10. Stone's theorem (30). 11. Weak topology, defined by a family of functions (33). 12. Topological product of spaces (34). 13. Metric spaces (36). 14. Compact sets in metric spaces (41). 15. Topological product of metric spaces (42).	
§ 3. Topological linear spaces	44
1. Definition of a topological linear space (44). 2. Closed subspaces in topological linear spaces (46). 3. Convex sets in locally convex spaces (46). 4. Defining a locally convex topology in terms of seminorms (47). 5. The case of a finite-dimensional space (50). 6. Continuous linear functionals (52). 7. Conjugate space (55). 8. Convex sets in a finite-dimensional space (58). 9. Convex sets in the conjugate space (58). 10. Cones (63). 11. Annihilators in the conjugate space (64). 12. Analytic vector-valued functions (66). 13. Complete locally convex spaces (67).	
§ 4. Normed spaces	67
1. Definition of a normed space (67). 2. Series in a normed space (72). 3. Quotient spaces of a Banach space (73). 4. Bounded linear operators (74). 5. Bounded linear functionals; conjugate space (78). 6. Compact (or completely continuous) operators (79). 7. Analytic vector-valued functions in a Banach space (80).	
§ 5. Hilbert space	82
1. Definition of Hilbert space (82). 2. Projection of a vector on a subspace (85). 3. Bounded linear functionals in Hilbert space (88). 4. Orthogonal systems of vectors in Hilbert space (90). 5. Orthogonal sum of subspaces (95). 6. Direct sum of Hilbert spaces (96). 7. Graph of an operator (97). 8. Closed operators; closure of an operator (97). 9. Adjoint operator (99). 10. The case of a bounded operator (103). 11. Generalization to operators in a Banach space (105). 12. Projection operators (106). 13. Reducibility (110). 14. Partially isometric operators (110). 15. Matrix representation of an operator (111).	
§ 6. Integration on locally compact spaces	114
1. Fundamental concepts; formulation of the problem (114). 2. Fundamental properties of the integral (114). 3. Extension of the integral to lower semi-continuous functions (115). 4. Upper integral of an arbitrary nonnegative real-valued function (118).	

5. Exterior measure of a set (119). 6. Equivalent functions (120). 7. The spaces \mathcal{L}^1 and L^1 (122). 8. Summable sets (125). 9. Measurable sets (128). 10. Measurable functions (129). 11. The real space L^2 (135). 12. The complex space L^2 (137). 13. The space L^∞ (137). 14. The positive and negative parts of a linear functional (137). 15. The Radon-Nikodým theorem (139). 16. The space conjugate to L^1 (140). 17. Complex measures (143). 18. Integrals on the direct product of spaces (144). 19. The integration of vector-valued and operator-valued functions (149).

CHAPTER II

FUNDAMENTAL CONCEPTS AND PROPOSITIONS IN THE THEORY OF NORMED ALGEBRAS

- § 7. **Fundamental algebraic concepts** 152
 1. Definition of a linear algebra (152). 2. Algebras with identity (153). 3. Center (156).
 4. Ideals (156). 5. The (Jacobson) radical (161). 6. Homomorphism and isomorphism
 of algebras (164). 7. Regular representations of algebras (165).
- § 8. **Topological algebras** 167
 1. Definition of a topological algebra (167). 2. Topological adjunction of the identity
 (169). 3. Algebras with continuous inverse (169). 4. Resolvents in an algebra with con-
 tinuous inverse (171). 5. Topological division algebras with continuous inverse (173).
 6. Algebras with continuous quasi-inverse (173).
- § 9. **Normed algebras** 174
 1. Definition of a normed algebra (174). 2. Adjunction of the identity (176). 3. The
 radical in a normed algebra (176). 4. Banach algebras with identity (177). 5. Resolvent
 in a Banach algebra with identity (179). 6. Continuous homomorphisms of normed
 algebras (179). 7. Regular representations of a normed algebra (180).
- § 10. **Symmetric algebras** 183
 1. Definition and simplest properties of a symmetric algebra (183). 2. Positive func-
 tionals (185). 3. Normed symmetric algebras (187). 4. Positive functionals in a sym-
 metric Banach algebra (188).

CHAPTER III

COMMUTATIVE NORMED ALGEBRAS

- § 11. **Realization of a commutative normed algebra in the form of an algebra of functions** . . 191
 1. Quotient algebra modulo a maximal ideal (191). 2. Functions on maximal ideals
 generated by elements of an algebra (192). 3. Topologization of the set of all maximal
 ideals (195). 4. The case of an algebra without identity (198). 5. System of generators
 of an algebra (199). 6. Analytic functions of algebra elements (200). 7. Wiener pairs
 of algebras (203). 8. Functions of several algebra elements; locally analytic functions
 (205). 9. Decomposition of an algebra into the direct sum of ideals (207). 10. Algebras
 with radical (207).
- § 12. **Homomorphism and isomorphism of commutative algebras** 208
 1. Uniqueness of the norm in a semisimple algebra (208). 2. The case of symmetric
 algebras (210).
- § 13. **Algebra (or Shilov) boundary** 210
 1. Definition and fundamental properties of the algebra boundary (210). 2. Extension
 of maximal ideals (212).

§ 14. Completely symmetric commutative algebras	215
1. Definition of a completely symmetric algebra (215). 2. Criterion for complete symmetry (215). 3. Application of Stone's theorem (216). 4. The algebra boundary of a completely symmetric algebra (218).	
§ 15. Regular algebras	218
1. Definition of a regular algebra (218). 2. Normal algebras of functions (219). 3. Structure space of an algebra (221). 4. Properties of regular algebras (222). 5. The case of an algebra without identity (226). 6. Sufficient condition that an algebra be regular (227). 7. Primary ideals (227).	
§ 16. Completely regular commutative algebras	228
1. Definition and simplest properties of a completely regular algebra (228). 2. Realization of completely regular commutative algebras (230). 3. Generalization to multi-normed algebras (236). 4. Symmetric subalgebras of the algebra $C(T)$ and compact extensions of the space T (237). 5. Antisymmetric subalgebras of the algebra $C(T)$ (238). 6. Subalgebras of the algebra $C(T)$ and certain problems in approximation theory (239).	

CHAPTER IV

REPRESENTATIONS OF SYMMETRIC ALGEBRAS

§ 17. Fundamental concepts and propositions in the theory of representations	242
1. Definitions and simplest properties of a representation (242). 2. Direct sum of representations (243). 3. Description of representations in terms of positive functionals (244). 4. Representations of completely regular commutative algebras; spectral theorem (248). 5. Spectral operators (256). 6. Irreducible representations (258). 7. Connection between vectors and positive functionals (259).	
§ 18. Embedding of a symmetric algebra in an algebra of operators	260
1. Regular norm (260). 2. Reduced algebras (261). 3. Minimal regular norm (264).	
§ 19. Indecomposable functionals and irreducible representations	256
1. Positive functionals, dominated by a given positive functional (266). 2. The algebra C_I (269). 3. Indecomposable positive functionals (270). 4. Completeness and approximation theorems (270).	
§ 20. Application to commutative symmetric algebras	274
1. Minimal regular norm in a commutative symmetric algebra (274). 2. Positive functionals in a commutative symmetric algebra (275). 3. Examples (278). 4. The case of a completely symmetric algebra (281).	
§ 21. Generalized Schur lemma	288
1. Canonical decomposition of an operator (288). 2. Fundamental theorem (290). 3. Application to direct sums of pairwise non-equivalent representations (292). 4. Application to representations which are multiples of a given irreducible representation (293).	
§ 22. Some representations of the algebra $\mathfrak{B}(\mathfrak{H})$	295
1. Ideals in the algebra $\mathfrak{B}(\mathfrak{H})$ (296). 2. The algebra I_0 and its representations (298). 3. Representations of the algebra $\mathfrak{B}(\mathfrak{H})$ (300).	

CHAPTER V
SOME SPECIAL ALGEBRAS

§ 23. Completely symmetric algebras	303
1. Definition and examples of completely symmetric algebras (303). 2. Spectrum (304). 3. Theorems on extensions (306). 4. Criterion for complete symmetry (312).	
§ 24. Completely regular algebras	314
1. Fundamental properties of completely regular algebras (314). 2. Realization of a completely regular algebra as an algebra of operators (316). 3. Quotient algebra of a completely regular algebra (319).	
§ 25. Dual algebras	320
1. Annihilator algebras and dual algebras (320). 2. Ideals in an annihilator algebra (322). 3. Semisimple annihilator algebras (325). 4. Simple annihilator algebras (330). 5. H^* -algebras (334). 6. Completely regular dual algebras (336).	
§ 26. Algebras of vector-valued functions	339
1. Definition of an algebra of vector-valued functions (339). 2. Ideals in an algebra of vector-valued functions (340). 3. Conditions for a vector-valued function to belong to an algebra (342). 4. The case of completely regular algebras (343). 5. Continuous analogue of the Schur lemma (351). 6. Structure space of a completely regular algebra (358).	

CHAPTER VI
GROUP ALGEBRAS

§ 27. Topological groups	361
1. Definition of a group (361). 2. Subgroups (362). 3. Definition and simplest properties of a topological group (363). 4. Invariant integrals and invariant measures on a locally compact group (364). 5. Existence of an invariant integral on a locally compact group (365).	
§ 28. Definition and fundamental properties of a group algebra	373
1. Definition of a group algebra (373). 2. Some properties of the group algebra (377).	
§ 29. Unitary representations of a locally compact group and their relationship with the representations of the group algebra	380
1. Unitary representations of a group (380). 2. Relationship between representations of a group and of the group algebra (381). 3. Completeness theorem (385). 4. Examples: a) Unitary representations of the group of linear transformations of the real line (386); b) Unitary representations of the proper Lorentz group (387); c) Example of a group algebra which is not completely symmetric (393).	
§ 30. Positive definite functions	398
1. Positive definite functions and their relationship with unitary representations (398). 2. Relationship of positive definite functions with positive functionals on a group algebra (400). 3. Regular sets (404). 4. Trigonometric polynomials on a group (407). 5. Spectrum (407).	
§ 31. Harmonic analysis on commutative locally compact groups	410
1. Maximal ideals of the group algebra of a commutative group; characters (410). 2. Group of characters (415). 3. Positive definite functions on a commutative group	

(416). 4. Inversion formula and Plancherel's theorem for commutative groups (418).
 5. Separation property of the set $[L^1 \cap P]$ (424). 6. Duality theorem (424). 7. Unitary representations of commutative groups (426). 8. Theorems of Tauberian type (427).
 9. The case of a compact group (432). 10. Spherical functions (433). 11. The generalized translation operation (435).

§ 32. Representations of compact groups 438
 1. The algebra $L^2(\mathfrak{G})$ (438). 2. Representations of a compact group (439). 3. Tensor product of representations (445). 4. Duality theorem for a compact group (446).

CHAPTER VII
 ALGEBRAS OF OPERATORS IN HILBERT SPACE

§ 33. Various topologies in the algebra $\mathfrak{B}(\mathfrak{H})$ 449
 1. Weak topology (449). 2. Strong topology (449). 3. Strongest topology (451). 4. Uniform topology (452).

§ 34. Weakly closed subalgebras of the algebra $\mathfrak{B}(\mathfrak{H})$ 452
 1. Fundamental concepts (452). 2. Principal identity (453). 3. Center (457). 4. Factorization (458).

§ 35. Relative equivalence 458
 1. Operators and subspaces adjoined to an algebra (458). 2. Fundamental lemma (459). 3. Definition of relative equivalence (460). 4. Comparison of closed subspaces (460). 5. Finite and infinite subspaces (463).

§ 36. Relative dimension 467
 1. Entire part of the ratio of two subspaces (467). 2. The case when a minimal subspace exists (468). 3. The case when a minimal subspace does not exist (469). 4. Existence and properties of relative dimension (470). 5. The range of the relative dimension; classification of factors (475). 6. Invariance of factor type under symmetric isomorphisms (477).

§ 37. Relative trace 478
 1. Definition of trace (478). 2. Properties of the trace (479). 3. Traces in factors of types (I_∞) and (II_∞) (484).

§ 38. Structure and examples of some types of factors 484
 1. The mapping $M \rightarrow M_{(\mathfrak{M})}$ (484). 2. Matrix description of factors of types (I) and (II) (486). 3. Description of factors of type (I) (488). 4. Structure of factors of type (II_∞) (490). 5. Example of a factor of type (II_1) (490). 6. Approximately finite factors of type (II_1) (493). 7. Relationship between the types of factors M and M' (493). 8. Relationship between symmetric and spatial isomorphisms (494). 9. Unbounded operators, adjoined to a factor of finite type (494).

§ 39. Unitary algebras and algebras with trace 494
 1. Definition of a unitary algebra (494). 2. Definition of an algebra with trace (494).
 3. Unitary algebras defined by the trace (495). 4. Canonical trace in a unitary algebra (495).

CHAPTER VIII
 DECOMPOSITION OF AN ALGEBRA OF OPERATORS
 INTO IRREDUCIBLE ALGEBRAS

§ 40. Formulation of the problem; canonical form of a commutative algebra of operators in Hilbert space	499
1. Formulation of the problem (499). 2. The separability lemma (501). 3. Canonical form of a commutative algebra (502).	
§ 41. Direct integral of Hilbert spaces; the decomposition of an algebra of operators into the direct integral of irreducible algebras	505
1. Direct integral of Hilbert spaces (505). 2. Decomposition of a Hilbert space into a direct integral with respect to a given commutative algebra R (508). 3. Decomposition with respect to a maximal commutative algebra; condition for irreducibility (513). 4. Decomposition of a unitary representation of a locally compact group into irreducible representations (517). 5. Central decompositions and factor representations (521). 6. Representations in a space with an indefinite metric (521).	
Appendix I Partially ordered sets and Zorn's lemma	523
Appendix II Borel spaces and Borel functions	523
Appendix III Analytic sets	525
Literature	532
Index	575