

# Contents

|   |     |
|---|-----|
| Preface .....   | ix  |
| <b>Chapter 1. Introduction</b> .....  | 1   |
| 1.1. The dynamical zeta functions .....   | 1   |
| 1.2. The motivations of the cohomological theory .....  | 4   |
| 1.2.1. Quantization of chaos .....  | 4   |
| 1.2.2. Uniform descriptions of the divisors of zeta functions .....   | 7   |
| 1.3. The contents of the book .....   | 13  |
| 1.3.1. Spectral theory on $X$ , Lefschetz formulas on $SX$ and<br>$\Gamma$ -invariant distributions on the ideal boundary $S^{n-1}$ ..... | 13  |
| 1.3.2. Harmonic currents and divisors of the zeta functions.<br>The main ideas .....  | 25  |
| 1.3.3. Harmonic currents and divisors of the zeta functions.<br>The results and the conjectures .....                                     | 30  |
| <b>Chapter 2. Preliminaries</b> .....   | 63  |
| 2.1. General notation .....   | 63  |
| 2.2. Lie theory related to the conformal group .....  | 63  |
| 2.3. Hyperbolic spaces as Riemannian manifolds<br>and symmetric spaces .....  | 67  |
| 2.4. $\mathfrak{n}^-$ -homology, $\mathfrak{n}^-$ -cohomology and Osborne's character formula .....                                       | 75  |
| 2.5. Induced representations and differential intertwining operators .....  | 76  |
| 2.6. The classification of the unitary irreducible representations<br>of the Lorentz group $SO(1, n)^\circ$ .....                         | 79  |
| <b>Chapter 3. Zeta Functions of the Geodesic Flow<br/>of Compact Locally Symmetric Manifolds</b> .....                                    | 87  |
| 3.1. Spectral theory of operators .....   | 88  |
| 3.2. The dynamical Lefschetz formula .....  | 103 |
| 3.3. Explicit formulas for the divisor in terms of<br>complexes on the ideal boundary .....   | 177 |
| 3.4. Patterson's conjecture .....   | 218 |
| <b>Chapter 4. Operators and Complexes</b> .....   | 231 |
| 4.1. Equivariant differential operators and equivariant differential<br>complexes for the twisted geodesic flows .....                    | 231 |
| 4.1.1. The de Rham complexes and the canonical complexes .....  | 231 |

|  |     |
|--|-----|
| 4.1.2. Geometry of the operators $d^-$ , $\delta^-$ , $D^+$ and $\Delta^+$ .....                               | 262 |
| 4.1.2.1. The operator $\square_p$ and the spaces $CC_{\lambda}^{(p,0)}(SY, \mathcal{V}_{\sigma})$ .....        | 264 |
| 4.1.2.2. The complexes on $CT_{\lambda-2p}^{(p,0)}(SY, \mathcal{V}_{\sigma})$ .....                            | 276 |
| 4.1.2.3. The Euler operator .....  | 281 |
| 4.1.2.4. More commutator relations .....   | 283 |
| 4.1.2.5. The operators $D_{\sigma}^+$ and $\square_{\sigma}$ .....   | 288 |
| 4.1.2.6. The operators $\delta^-(\Omega_{\sigma}^- \wedge)$ and $\Omega_{\sigma}^- \wedge \delta^-$ .....      | 290 |
| 4.1.2.7. The spaces $S_{\lambda}^{(p,0)}(SY, \mathcal{V}_{\sigma})$ .....                                      | 292 |
| 4.2. The Bruhat and Iwasawa models .....   | 296 |
| 4.2.1. The Bruhat models of the operators $D^+$ and $\square$ .....  | 297 |
| 4.2.2. The Iwasawa models of the operators $D^+$ and $\square$ .....   | 315 |
| <b>Chapter 5. The Verma Complexes on <math>SY</math> and <math>SX</math></b> .....                             | 331 |
| 5.1. The Bruhat models of the Verma complexes on $SY$ .....  | 331 |
| 5.2. The Iwasawa models of the Verma complexes on $SY$ .....   | 343 |
| 5.3. The Verma complexes on $SX$ .....   | 360 |
| <b>Chapter 6. Harmonic Currents and Canonical Complexes</b> .....  | 373 |
| 6.1. Equivariant Hodge decomposition of $CC_{\lambda-2p}^{(p,0)}(SY)$ for $\lambda \notin -\mathbb{N}_0$ ..... | 374 |
| 6.2. Equivariant right parametrices of $D^+$ and $\delta^-$ for $\lambda \in -\mathbb{N}_0$ .....              | 392 |
| 6.3. Hodge decomposition of $C\hat{C}_{(p,0)}^{\lambda}(SX)$ for $\lambda \notin -\mathbb{N}_0$ .....          | 410 |
| 6.4. Hodge decomposition of $C\hat{C}_{(p,0)}^{\lambda}(SX)$ for $\lambda \in -\mathbb{N}_0$ .....             | 412 |
| 6.5. The system $\square^-\omega = 0$ , $\hat{\square}\omega = 0$ and exotic currents .....                    | 432 |
| 6.6. The functional equation as an index formula .....   | 452 |
| <b>Chapter 7. Divisors and Harmonic Currents</b> .....   | 469 |
| 7.1. The divisor of the Selberg zeta function .....  | 469 |
| 7.2. The divisor of the Ruelle zeta function .....   | 480 |
| 7.3. Harmonic currents which are constant on the leaves of $\mathcal{P}^-$ .....                               | 486 |
| 7.4. The Ruelle zeta functions of the geodesic flow of $\Gamma \backslash \mathbb{H}^4$ .....                  | 493 |
| <b>Chapter 8. Further Developments and Open Problems</b> .....   | 519 |
| 8.1. The divisor of $Z_S$ for convex-cocompact groups .....  | 519 |
| 8.1.1. Scattering operators, extension operators and invariant currents on the limit set .....                 | 525 |
| 8.1.2. $\Gamma$ -cohomology of holomorphic families of hyperfunctions on the limit set .....                   | 569 |
| 8.1.3. The embedded case .....   | 585 |
| 8.1.4. $\Gamma$ -cohomology and harmonic currents .....  | 607 |
| 8.2. Miscellaneous problems and comments .....   | 624 |

|                                 |   |     |
|---------------------------------|---|-----|
| 8.2.1.                          | The relations between the various definitions of twisted Selberg zeta functions .....                       | 624 |
| 8.2.2.                          | Dynamical theta functions .....   | 625 |
| 8.2.3.                          | Zeta functions and zeta-regularized determinants .....  | 637 |
| 8.2.4.                          | Closed ranges in the tangential complex of the stable foliation .....                                       | 639 |
| 8.2.5.                          | The spaces $\hat{CV}_{(n-1,0)}^\lambda(SX)$ and the operators $\mathcal{L}_0(s)$ .....                      | 640 |
| 8.2.6.                          | Hodge decompositions .....  | 644 |
| 8.2.7.                          | The equation $d^-\omega = \theta$ in the twisted case .....   | 647 |
| 8.2.8.                          | Patterson's conjecture as a fixed point formula .....   | 650 |
| 8.2.9.                          | Topological singularities and group cohomology .....  | 652 |
| 8.2.10.                         | Meromorphic extension of Selberg zeta functions and smoothness of $\mathcal{P}^\pm$ .....                   | 655 |
| 8.2.11.                         | Zeta functions of the geodesic flow of rank one spaces .....  | 656 |
| 8.2.12.                         | Lefschetz formulas and zeta functions for flows associated to locally symmetric spaces of higher rank ..... | 657 |
| 8.2.13.                         | Zeta functions for negative curvature spaces .....  | 661 |
| 8.2.14.                         | Lefschetz fixed point formulas for foliations .....   | 666 |
| 8.3.                            | Some historical comments .....  | 667 |
| <b>Chapter 9.</b>               | <b>A Summary of Important Formulas</b> .....  | 673 |
| <b>Bibliography</b> .....       | 687   |     |
| <b>Index of Equations</b> ..... | 703   |     |
| <b>Index</b> .....              | 707   |     |