Contents

Pre	Preface to Volume III Contents of Volume I Contents of Volume II		'
Cor			X۱
Cor			xix
PA	RT	D LIE SUPERALGEBRAS, LIE SUPERGROUPS AND THEIR APPLICATIONS	1
20	In	troduction to Superalgebras and Supermatrices	;
	1 2 3 4	The notion of grading Associative superalgebras Grassmann algebras Supermatrices	15 15
21	General Properties of Lie Superalgebras		21
	1 2 3	Lie superalgebras introduced Definitions and immediate consequences Subalgebras, direct sums, and homomorphisms of Lie superalgebras	21 22 31
	4 5	Graded representations of Lie superalgebras The adjoint representation and the Killing form of a Lie superalgebra	35 41
22	Sı	uperspace and Lie Supergroups	45
	1	Grassmann variables as coordinates	45

x CONTENTS

	2	Analysis on superspace	46
		(a) The superspace $\mathbb{R}\mathrm{B}_{L}^{m,n}$	46
		(b) Differentiable functions on $\mathbb{R}\mathbf{B}_{L}^{m,n}$	50
		(c) Superanalytic and superdifferentiable functions on $\mathbb{R}B_L^{m,n}$	56
		(d) Differentiation of supermatrices	65
	3	Linear Lie supergroups	68
		(a) The definition of a Lie supergroup and its associated "super" Lie algebra	68
		(b) The relationship between Lie superalgebras and Lie supergroups	75
23	Th	ne Poincaré Superalgebras and Supergroups	79
	1	Introduction	79
	2	The $N = 1$, $D = 4$ Poincaré superalgebra and supergroup	80
	-	(a) The Lie superalgebra extension of the Poincaré algebra	80
		(b) The two-component formulation for the Poincaré	
		superalgebra	88
		(c) The $N = 1$, $D = 4$ Poincaré supergroup	92
		(d) Action of the $N = 1$, $D = 4$ Poincaré supergroup on	
		superspace	100
	3	Extended Poincaré superalgebras and Poincaré supergroups	
		for $D=4$	107
	4	The Poincaré superalgebras and supergroups for Minkowski	
		space-times of general dimension D	118
		(a) The unextended Poincaré superalgebras and supergroups for	
		general dimension D	118
		(b) The extended Poincaré superalgebras and supergroups for	
		general dimension D	123
	5	Irreducible representations of the unextended $D=4$	
		Poincaré superalgebra	126
		(a) Irreducible representations of the unextended $D = 4$ Poincaré	
		superalgebra corresponding to $M > 0$	127
		(b) Irreducible representations of the unextended $D = 4$ Poincaré	
		superalgebra corresponding to $M = 0$	135
	6	Irreducible representations of the extended $D = 4$ Poincaré	
		superalgebras	138
		(a) Irreducible representations of the N-extended Poincaré	
		superalgebra corresponding to $M > 0$	138
		(b) Irreducible representations of the N-extended Poincaré	
		superalgebra corresponding to $M=0$	145
	7	Irreducible representations of the Poincaré superalgebras for	
		general space-time dimensions	149
		(a) Irreducible representations of the unextended	
		D-dimensional Poincaré superalgebras	149
		(b) Irreducible representations of the extended D-dimensional	
		Poincaré superalgebras	155

CONTENTS xi

24	Po	pincaré Supersymmetric Fields	161
	1	Supersymmetric field theory	161
	2	Supersymmetric multiplets	162
		(a) The chiral multiplet in component form	162
		(b) The Wess-Zumino model in component form	170
		(c) The general multiplet in component form	174
	3	Superfields	181
		(a) The scalar superfield	182
		(b) The chiral superfields	185
		(c) Superfield formulation of the action of the Wess-Zumino model	190
	4	Supersymmetric gauge theories	192
	•	(a) The component formulation of super-QED	192
		(b) Super-QED in a superfield formulation	201
		(c) Supersymmetric Yang–Mills theories in a superfield	
		formulation	206
	5	Spontaneous symmetry breaking	214
25	Si	mple Lie Superalgebras	219
	1	An outline of the presentation	219
	2	The definition of a simple Lie superalgebra and some immediate	
	-	consequences	220
	3	Classical simple Lie superalgebras	223
	•	(a) Definition and basic theorems	223
		(b) Basic classical simple complex Lie superalgebras	230
		(c) Simple roots, Cartan matrices, generalized Dynkin diagrams and the Weyl group	238
	4	Graded representations of basic classical simple complex Lie	250
	4	superalgebras	245
		(a) Weights and highest weights	245
		(b) "Typical" and "atypical" irreducible representations	253
		(c) Casimir operators and indices of representations	260
	5	The classical simple real Lie superalgebras	267
	6	The conformal, de Sitter and anti-de Sitter superalgebras	275
	Ü	The comornia, de Sitter and anti-de Sitter superargeoras	2,0
PA	RT	E INFINITE-DIMENSIONAL LIE ALGEBRAS AND	
		SUPERALGEBRAS AND THEIR APPLICATIONS	279
26	Ti	ne Structure of Kac–Moody Algebras	281
	1	Introduction to infinite-dimensional Lie algebras	281
	2	Construction of Kac-Moody algebras	282
	3	Properties of general Kac–Moody algebras	289
	4	Types of complex Kac-Moody algebras	299

xii CONTENTS

	5	Affine Kac-Moody algebras (a) General deductions	307
		(b) Construction of the complex untwisted affine Kac-Moody	307
		algebras	310
		(c) Construction of the complex twisted affine Kac-Moody algebras	318
		(d) Root lattices and the Weyl group	329
		(e) The compact real form of a complex affine Kac-Moody	
	_	algebra	334
	6	Kac-Moody superalgebras	335
27	R	epresentations of Kac–Moody Algebras	339
	1	Highest weight representations of general Kac-Moody algebras	339
	2	Highest weight representations of affine Kac-Moody algebras	342
	3 4	Character formulae	351
	7	The vertex construction of the basic representation of a simply laced untwisted affine Kac-Moody algebra	353
	5	Representations of untwisted affine Kac-Moody algebras in terms	555
		of fermion creation and annihilation operators	362
28	Tŀ	ne Virasoro Algebra and Superalgebras	369
	1	The conformal algebras	369
	2	Representations of the Virasoro algebra	373
	3	Some constructions of highest weight representations of the	
		Virasoro algebra	376
	4	Virasoro superalgebras	382
29	ΑI	gebraic Aspects of the Theory of Strings and Superstrings	389
	1	Introduction	389
	2	The bosonic string	389
		(a) The Lagrangian density for the bosonic string	389
		(b) The classical open string(c) Light-cone quantization of the open bosonic string	394
		(d) Covariant quantization of the open bosonic string	398 402
		(e) The closed bosonic string	406
	3	The spinning string of Ramond, Neveu and Schwarz	411
	4	The superstring of Green and Schwarz	417
		(a) The light-cone Lagrangian density	417
		(b) Light-cone quantization of the open superstring	423
		(c) Light-cone quantization of the closed superstring	428
	5	(d) Torus compactification, the field theory limit and interactions The heterotic string	431 437
	-	(a) The right-moving and left-moving modes	437

CONTENTS xiii

	(b) The appearance of the $E_8 \oplus E_8$ algebras and the Spin(32)/Z ₂ group	441
6	Further developments	446
APPEI	NDICES	449
Apper	ndix K Proofs of Certain Theorems on Supermatrices and Lie Superalgebras	451
1	Proofs of Theorems I and IV of Chapter 20, Section 4	451
2	Proof of Theorem I of Chapter 21, Section 4	457
3	Proof of Theorem III of Chapter 21, Section 5	459
4 5	Proofs of Theorems II, III, IV and V of Chapter 25, Section 2 Proofs of Theorems VI, VII, VIII, IX, XVI, XX, XXII and XXIII of	460
5	Chapter 25, Sections $3(a)$, $3(b)$ and $3(c)$	463
6	Proofs of Theorems III and IV of Chapter 25, Section 4(a)	470
Apper	dix L Clifford Algebras	475
1	The Clifford algebras of D-dimensional space-times	475
2	Irreducible representations for the case in which D is even	477
	(a) Explicit expressions for the matrices	477
	(b) Chirality of the representation	480
	(c) Reality of the spinor representation of $so(D-1,1)$	482
_	(d) The generalized charge conjugation matrices	486
3	Irreducible representations for the case in which <i>D</i> is odd	489 489
	(a) Explicit expressions for the matrices(b) Non-chirality of the representations	490
	(c) Reality of the spinor representations of $so(D-1, 1)$	491
	(d) The generalized charge conjugation matrices	493
4		100
•	Minkowski Clifford algebra and those of the $(D-2)$ -	
	dimensional Euclidean Clifford algebra	496
5	A matrix identity for the $D = 4$ Minkowski Clifford algebra	501
Apper	ndix M Properties of the Classical Simple Complex Lie Superalgebras	503
1	The basic type I classical simple complex Lie superalgebras $A(r/s)$, $r>s\geqslant 0$	503
2		500
	$A(r/r), r \geqslant 1$	509
3	The basic type II classical simple complex Lie superalgebras $B(r/s)$, $r \ge 0$, $s \ge 1$	513

xiv CONTENTS

4	, the second of the second of (a),	F04
5	$s \ge 2$ The basic type II classical simple complex Lie superalgebras $D(r/s)$	521
3	$r \ge 2$, $s \ge 1$	525
6		020
·	$D(2/1; \alpha)$, with α a complex parameter taking all values other than	
	$0, -1$ and ∞	532
7		537
8	The basic type II classical simple complex Lie superalgebra $G(3)$	542
9	,,	042
·	$P(r), r \ge 2$	545
10	• • •	0.0
	$Q(r), r \ge 2$	547
Annon	div Al Droportion of the Common Affine Was Based.	
Appen	dix N Properties of the Complex Affine Kac-Moody Algebras	551
	Aigebias	551
1		551
2	, , , , , , , , , , , , , , , , , , , ,	552
3		553
4		554
5		556
6	, 5	558
7	,	559
8	, , ,	560
9		561
10		562
11	, ,	563
12	, 0 21	564
13	, , , , , , , , , , , , , , , , , , , ,	566
14	, 5	568
15		570
16	The complex twisted affine Kac–Moody algebra $D_4^{(3)}$	571
Appen	dix O Proofs of Certain Theorems on Kac-Moody and	
	Virasoro Algebras	575
1	Proofs of Theorems I, III and IV of Chapter 27, Section 2	575
2	Proofs of Theorems I and II of Chapter 27, Section 4	579
3	Proof of Theorem I of Chapter 27, Section 5	589
4	Proofs of Theorems I and II of Chapter 28, Section 3	591
•	Thousand I and II of Onapter 20, occitor 5	551
. .		.
Keteren	ces for Volume III	599
Index		617