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$$z \mapsto \frac{az + b}{cz + d}, \text{ where } ad - bc \neq 0.$$

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Conjugate geometric transformations have the same geometric structure. Normal subgroups are formed from a union of conjugacy classes.
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The set of transformations of the form

$$x \mapsto \frac{ax + b}{cx + d}, \text{ where } ad - bc \neq 0,$$

is a homomorphic image of the group $GL(2, F)$.
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