

Contents

Preface	xiii
Volume I: Structure of strongly quasithin \mathcal{K}-groups	1
Introduction to Volume I	3
0.1. Statement of Main Results	3
0.2. An overview of Volume I	5
0.3. Basic results on finite groups	7
0.4. Semisimple quasithin and strongly quasithin \mathcal{K} -groups	7
0.5. The structure of SQTK-groups	7
0.6. Thompson factorization and related notions	8
0.7. Minimal parabolics	10
0.8. Pushing up	10
0.9. Weak closure	11
0.10. The amalgam method	11
0.11. Properties of \mathcal{K} -groups	12
0.12. Recognition theorems	13
0.13. Background References	15
Chapter A. Elementary group theory and the known quasithin groups	19
A.1. Some standard elementary results	19
A.2. The list of quasithin \mathcal{K} -groups: Theorems A, B, and C	32
A.3. A structure theory for Strongly Quasithin \mathcal{K} -groups	41
A.4. Signalizers for groups with $\mathbf{X} = \mathbf{O}^2(\mathbf{X})$	56
A.5. An ordering on $\mathcal{M}(T)$	61
A.6. A group-order estimate	64
Chapter B. Basic results related to Failure of Factorization	67
B.1. Representations and FF-modules	67
B.2. Basic Failure of Factorization	74
B.3. The permutation module for A_n and its FF*-offenders	83
B.4. \mathbf{F}_2 -representations with small values of q or \hat{q}	85
B.5. FF-modules for SQTK-groups	98
B.6. Minimal parabolics	112
B.7. Chapter appendix: Some details from the literature	118
Chapter C. Pushing-up in SQTK-groups	121
C.1. Blocks and the most basic results on pushing-up	121
C.2. More general pushing up in SQTK-groups	143
C.3. Pushing up in unconstrained 2-locals	148

C.4.	Pushing up in constrained 2-locals	151
C.5.	Finding a common normal subgroup	154
C.6.	Some further pushing up theorems	164
Chapter D.	The <i>qrc</i> -lemma and modules with $\hat{q} \leq 2$	171
D.1.	Stellmacher's <i>qrc</i> -Lemma	171
D.2.	Properties of q and \hat{q} : $\mathcal{R}(G, V)$ and $\mathcal{Q}(G, V)$	177
D.3.	Modules with $\hat{\mathbf{q}} \leq \mathbf{2}$	192
Chapter E.	Generation and weak closure	209
E.1.	\mathcal{E} -generation and the parameter $\mathbf{n}(\mathbf{G})$	209
E.2.	Minimal parabolics under the SQTk-hypothesis	215
E.3.	Weak Closure	230
E.4.	Values of \mathbf{a} for \mathbf{F}_2 -representations of SQTk-groups.	240
E.5.	Weak closure and higher Thompson subgroups	242
E.6.	Lower bounds on $\mathbf{r}(\mathbf{G}, \mathbf{V})$	244
Chapter F.	Weak BN-pairs and amalgams	259
F.1.	Weak BN-pairs of rank 2	259
F.2.	Amalgams, equivalences, and automorphisms	264
F.3.	Paths in rank-2 amalgams	269
F.4.	Controlling completions of Lie amalgams	273
F.5.	Identifying $\mathbf{L}_4(\mathbf{3})$ via its $\mathbf{U}_4(\mathbf{2})$ -amalgam	299
F.6.	Goldschmidt triples	304
F.7.	Coset geometries and amalgam methodology	310
F.8.	Coset geometries with $\mathbf{b} > \mathbf{2}$	315
F.9.	Coset geometries with $\mathbf{b} > \mathbf{2}$ and $\mathbf{m}(\mathbf{V}_1) = \mathbf{1}$	317
Chapter G.	Various representation-theoretic lemmas	327
G.1.	Characterizing direct sums of natural $\mathbf{SL}_n(\mathbf{F}_{2^a})$ -modules	327
G.2.	Almost-special groups	332
G.3.	Some groups generated by transvections	337
G.4.	Some subgroups of $\mathbf{Sp}_4(\mathbf{2}^n)$	338
G.5.	\mathbf{F}_2 -modules for \mathbf{A}_6	342
G.6.	Modules with $\mathbf{m}(\mathbf{G}, \mathbf{V}) \leq \mathbf{2}$	345
G.7.	Small-degree representations for some SQTk-groups	346
G.8.	An extension of Thompson's dihedral lemma	349
G.9.	Small-degree representations for more general SQTk-groups	351
G.10.	Small-degree representations on extraspecial groups	357
G.11.	Representations on extraspecial groups for SQTk-groups	364
G.12.	Subgroups of $\mathbf{Sp}(\mathbf{V})$ containing transvections on hyperplanes	370
Chapter H.	Parameters for some modules	377
H.1.	$\Omega_4^\epsilon(\mathbf{2}^n)$ on an orthogonal module of dimension $4\mathbf{n}$ ($\mathbf{n} > \mathbf{1}$)	378
H.2.	$\mathbf{SU}_3(\mathbf{2}^n)$ on a natural $6\mathbf{n}$ -dimensional module	378
H.3.	$\mathbf{Sz}(\mathbf{2}^n)$ on a natural $4\mathbf{n}$ -dimensional module	379
H.4.	$(\mathbf{S})\mathbf{L}_3(\mathbf{2}^n)$ on modules of dimension 6 and 9	379
H.5.	7-dimensional permutation modules for $\mathbf{L}_3(\mathbf{2})$	385
H.6.	The 21-dimensional permutation module for $\mathbf{L}_3(\mathbf{2})$	386
H.7.	$\mathbf{Sp}_4(\mathbf{2}^n)$ on natural $4\mathbf{n}$ plus the conjugate $4\mathbf{n}^t$.	388

H.8.	\mathbf{A}_7 on $4 \oplus \bar{4}$	389
H.9.	$\mathbf{Aut}(\mathbf{L}_n(\mathbf{2}))$ on the natural \mathbf{n} plus the dual \mathbf{n}^*	389
H.10.	A foreword on Mathieu groups	392
H.11.	\mathbf{M}_{12} on its 10-dimensional module	392
H.12.	$3\mathbf{M}_{22}$ on its 12-dimensional modules	393
H.13.	Preliminaries on the binary code and cocode modules	395
H.14.	Some stabilizers in Mathieu groups	396
H.15.	The cocode modules for the Mathieu groups	398
H.16.	The code modules for the Mathieu groups	402
Chapter I.	Statements of some quoted results	407
I.1.	Elementary results on cohomology	407
I.2.	Results on structure of nonsplit extensions	409
I.3.	Balance and 2-components	414
I.4.	Recognition Theorems	415
I.5.	Characterizations of $L_4(2)$ and $Sp_6(2)$	418
I.6.	Some results on TI-sets	424
I.7.	Tightly embedded subgroups	425
I.8.	Discussion of certain results from the Bibliography	428
Chapter J.	A characterization of the Rudvalis group	431
J.1.	Groups of type Ru	431
J.2.	Basic properties of groups of type Ru	432
J.3.	The order of a group of type Ru	438
J.4.	A ${}^2\mathbf{F}_4(\mathbf{2})$ -subgroup	440
J.5.	Identifying G as Ru	445
Chapter K.	Modules for SQTK-groups with $\hat{q}(G, V) \leq 2$.	451
	Notation and overview of the approach	451
K.1.	Alternating groups	452
K.2.	Groups of Lie type and odd characteristic	453
K.3.	Groups of Lie type and characteristic 2	453
K.4.	Sporadic groups	457
	Bibliography and Index	461
	Background References Quoted (Part 1: also used by GLS)	463
	Background References Quoted (Part 2: used by us but not by GLS)	465
	Expository References Mentioned	467
	Index	471
	Volume II: Main Theorems; the classification of simple QTKE- groups	479
	Introduction to Volume II	481
	0.1. Statement of Main Results	481

0.2. Context and History	483
0.3. An Outline of the Proof of the Main Theorem	487
0.4. An Outline of the Proof of the Even Type Theorem	495
Part 1. Structure of QTKE-Groups and the Main Case Division	497
Chapter 1. Structure and intersection properties of 2-locals	499
1.1. The collection \mathcal{H}^e	499
1.2. The set $\mathcal{L}^*(G, T)$ of nonsolvable uniqueness subgroups	503
1.3. The set $\Xi^*(G, T)$ of solvable uniqueness subgroups of G	508
1.4. Properties of some uniqueness subgroups	514
Chapter 2. Classifying the groups with $ \mathcal{M}(T) = 1$	517
2.1. Statement of main result	518
2.2. Bender groups	518
2.3. Preliminary analysis of the set Γ_0	521
2.4. The case where Γ_0^e is nonempty	527
2.5. Eliminating the shadows with Γ_0^e empty	550
Chapter 3. Determining the cases for $L \in \mathcal{L}_f^*(G, T)$	571
3.1. Common normal subgroups, and the <i>qrc</i> -lemma for QTKE-groups	571
3.2. The Fundamental Setup, and the case division for $\mathcal{L}_f^*(G, T)$	578
3.3. Normalizers of uniqueness groups contain $N_G(T)$	585
Chapter 4. Pushing up in QTKE-groups	605
4.1. Some general machinery for pushing up	605
4.2. Pushing up in the Fundamental Setup	608
4.3. Pushing up $L_2(2^n)$	613
4.4. Controlling suitable odd locals	619
Part 2. The treatment of the Generic Case	627
Chapter 5. The Generic Case: $L_2(2^n)$ in \mathcal{L}_f and $n(H) > 1$	629
5.1. Preliminary analysis of the $L_2(2^n)$ case	630
5.2. Using weak BN-pairs and the Green Book	646
5.3. Identifying rank 2 Lie-type groups	658
Chapter 6. Reducing $\mathbf{L}_2(2^n)$ to $\mathbf{n} = 2$ and V orthogonal	663
6.1. Reducing $\mathbf{L}_2(2^n)$ to $\mathbf{L}_2(4)$	663
6.2. Identifying M_{22} via $L_2(4)$ on the natural module	679
Part 3. Modules which are not FF-modules	693
Chapter 7. Eliminating cases corresponding to no shadow	695
7.1. The cases which must be treated in this part	696
7.2. Parameters for the representations	697
7.3. Bounds on w	698
7.4. Improved lower bounds for r	699
7.5. Eliminating most cases other than shadows	700
7.6. Final elimination of $\mathbf{L}_3(2)$ on $\mathbf{3} \oplus \bar{\mathbf{3}}$	701
7.7. mini-Appendix: $r > 2$ for $\mathbf{L}_3(2).2$ on $\mathbf{3} \oplus \bar{\mathbf{3}}$	703

Chapter 8. Eliminating shadows and characterizing the J_4 example	711
8.1. Eliminating shadows of the Fischer groups	711
8.2. Determining local subgroups, and identifying J_4	714
8.3. Eliminating $L_3(2) \wr 2$ on 9	723
Chapter 9. Eliminating $\Omega_4^+(2^n)$ on its orthogonal module	729
9.1. Preliminaries	729
9.2. Reducing to $n = 2$	730
9.3. Reducing to $n(H) = 1$	732
9.4. Eliminating $n(H) = 1$	735
Part 4. Pairs in the FSU over F_{2^n} for $n > 1$.	739
Chapter 10. The case $L \in \mathcal{L}_f^*(G, T)$ not normal in M .	741
10.1. Preliminaries	741
10.2. Weak closure parameters and control of centralizers	742
10.3. The final contradiction	755
Chapter 11. Elimination of $L_3(2^n)$, $Sp_4(2^n)$, and $G_2(2^n)$ for $n > 1$	759
11.1. The subgroups $N_G(V_i)$ for T -invariant subspaces V_i of V	760
11.2. Weak-closure parameter values, and $\langle V^{N_G(V_1)} \rangle$	766
11.3. Eliminating the shadow of $L_4(q)$	770
11.4. Eliminating the remaining shadows	775
11.5. The final contradiction	778
Part 5. Groups over F_2	785
Chapter 12. Larger groups over F_2 in $\mathcal{L}_f^*(G, T)$	787
12.1. A preliminary case: Eliminating $L_n(2)$ on $n \oplus n^*$	787
12.2. Groups over F_2 , and the case V a TI-set in G	794
12.3. Eliminating A_7	807
12.4. Some further reductions	812
12.5. Eliminating $L_5(2)$ on the 10-dimensional module	816
12.6. Eliminating A_8 on the permutation module	822
12.7. The treatment of \hat{A}_6 on a 6-dimensional module	838
12.8. General techniques for $L_n(2)$ on the natural module	849
12.9. The final treatment of $L_n(2)$, $n = 4, 5$, on the natural module	857
Chapter 13. Mid-size groups over F_2	865
13.1. Eliminating $L \in \mathcal{L}_f^*(G, T)$ with $L/O_2(L)$ not quasisimple	865
13.2. Some preliminary results on A_5 and A_6	876
13.3. Starting mid-sized groups over F_2 , and eliminating $U_3(3)$	884
13.4. The treatment of the 5-dimensional module for A_6	896
13.5. The treatment of A_5 and A_6 when $\langle V_3^{G_1} \rangle$ is nonabelian	915
13.6. Finishing the treatment of A_5	926
13.7. Finishing the treatment of A_6 when $\langle V^{G_1} \rangle$ is nonabelian	935
13.8. Finishing the treatment of A_6	946
13.9. Chapter appendix: Eliminating the A_{10} -configuration	969
Chapter 14. $L_3(2)$ in the FSU, and $L_2(2)$ when $\mathcal{L}_f(G, T)$ is empty	975

14.1.	Preliminary results for the case $\mathcal{L}_f(\mathbf{G}, \mathbf{T})$ empty	975
14.2.	Starting the $\mathbf{L}_2(\mathbf{2})$ case of \mathcal{L}_f empty	981
14.3.	First steps; reducing $\langle \mathbf{V}^{\mathbf{G}_1} \rangle$ nonabelian to extraspecial	989
14.4.	Finishing the treatment of $\langle \mathbf{V}^{\mathbf{G}_1} \rangle$ nonabelian	1005
14.5.	Starting the case $\langle \mathbf{V}^{\mathbf{G}_1} \rangle$ abelian for $\mathbf{L}_3(\mathbf{2})$ and $\mathbf{L}_2(\mathbf{2})$	1013
14.6.	Eliminating $\mathbf{L}_2(\mathbf{2})$ when $\langle \mathbf{V}^{\mathbf{G}_1} \rangle$ is abelian	1020
14.7.	Finishing $\mathbf{L}_3(\mathbf{2})$ with $\langle \mathbf{V}^{\mathbf{G}_1} \rangle$ abelian	1042
14.8.	The QTKE-groups with $\mathcal{L}_f(\mathbf{G}, \mathbf{T}) \neq \emptyset$	1078
Part 6.	The case $\mathcal{L}_f(G, T)$ empty	1081
Chapter 15.	The case $\mathcal{L}_f(\mathbf{G}, \mathbf{T}) = \emptyset$	1083
15.1.	Initial reductions when $\mathcal{L}_f(\mathbf{G}, \mathbf{T})$ is empty	1083
15.2.	Finishing the reduction to $\mathbf{M}_f/\mathbf{C}_{\mathbf{M}_f}(\mathbf{V}(\mathbf{M}_f)) \simeq \mathbf{O}_4^+(\mathbf{2})$	1104
15.3.	The elimination of $\mathbf{M}_f/\mathbf{C}_{\mathbf{M}_f}(\mathbf{V}(\mathbf{M}_f)) = \mathbf{S}_3$ wr \mathbf{Z}_2	1120
15.4.	Completing the proof of the Main Theorem	1155
Part 7.	The Even Type Theorem	1167
Chapter 16.	Quasithin groups of even type but not even characteristic	1169
16.1.	Even type groups, and components in centralizers	1169
16.2.	Normality and other properties of components	1173
16.3.	Showing L is standard in G	1177
16.4.	Intersections of $\mathbf{N}_{\mathbf{G}}(\mathbf{L})$ with conjugates of $\mathbf{C}_{\mathbf{G}}(\mathbf{L})$	1182
16.5.	Identifying \mathbf{J}_1 , and obtaining the final contradiction	1194
Bibliography and Index		1205
Background References Quoted		
	(Part 1: also used by GLS)	1207
Background References Quoted		
	(Part 2: used by us but not by GLS)	1209
Expository References Mentioned		1211
Index		1215