Contents

Preface vii

Part I Background Material 1

1. Rings and Modules 3

1.1. Notation and terminology 3
1.2. Preliminary results 8
1.3. Artinian and noetherian modules and rings 13
1.4. Semisimple modules 22
1.5. The radical and socle of modules and rings 26
1.5.A. The radical and socle of modules 26
1.5.B. The Jacobson radical 31
1.5.C. The Jacobson radical and idempotents 39
1.6. Idempotent lifting theory 42
1.6.A. General results 42
1.6.B. Semiregular rings 46
1.6.C. Algebras over complete rings 52
1.7. Azumaya's theorems 55
1.8. Local rings 60
1.9. Endomorphism algebras 61
1.10. Strongly indecomposable modules 65
1.10.A. Basic properties and characterizations 65
1.10.B. Azumaya's decomposition theorem 70
1.10.C. The Krull-Schmidt theorem 71
1.11. Direct decompositions and blocks 73
1.12. Matrix rings 80

2. Artinian and Semilocal Rings 89

2.1. Semiprimitive artinian rings 89
2.2. Semilocal rings 92
2.3. Artinian rings 100
2.3.A. A characterization 100

2.3.B. Principal indecomposable modules 101
2.3.C. Blocks and Cartan matrices 102
2.3.D. The Loewy and socle series 105
2.4. Representations of algebras 108

3. Homological Algebra 113

3.1. Tensor products of modules 113
3.2. Tensor product of algebras 121
3.3. Flat modules 126
3.4. Properties of $Hom_R(V,W)$ 132
3.5. Change of coefficient rings 139
3.6. Projective modules 141
3.6.A. Basic properties 141
3.6.B. Generators and progenerators 148
3.6.C. Endomorphism rings 152
3.7. Pushouts and pullbacks 155
3.8. Injective modules 158
3.8.A. Basic properties 158
3.8.B. Injective hulls 163
3.8.C. Endomorphism rings 169
3.9. Dual modules, bilinear forms and Frobenius algebras 171
3.9.A. Dual modules 171
3.9.B. Bilinear forms 178
3.9.C. Frobenius algebras 184

4. Restriction, Induction and Coinduction 191

4.1. General information 191
4.2. Universal characterizations 194
4.3. The splitting of canonical homomorphisms 196
4.4. Another characterization of induced modules 201
4.5. Induction and semisimplicity 202
4.6. Annihilators of induced modules 204
4.7. Exact sequences of induced modules 206
4.8. Normal subalgebras 208
4.9. Induction and relative projectivity 212
4.10. Coinduction and relative injectivity 218
4.11. Relative injective modules for algebras 220

5. Semiperfect Rings 225

5.1. Projective covers 225
5.2. Characterizations and fundamental properties 233

5.3. Some classes of semiperfect rings 239
5.4. Block decompositions 242
5.5. Properties of idempotents 246
5.6. Semiperfect endomorphism rings 249
5.7. Basic rings 255
5.8. Points in semiperfect rings 257

6. Complexes, Homology and Resolutions 261

6.1. Notation and terminology 261
6.2. Fundamental properties of complexes 264
6.3. Homotopy 270
6.4. Resolutions 275
6.5. Categories and functors 285
6.6. Universal functors and sattelites 291
6.7. Derived functors 302
6.8. *Ext* and *Tor* 311
6.9. Ext^1 and extensions 321

7. Heller Operators 327

7.1. Heller operators 327
7.2. Minimal resolutions 330
7.3. Self-injective artinian rings 336
7.4. Projective homomorphisms 341
7.5. Heller operators and dual modules 348

8. Group Algebras 351

8.1. Definitions and elementary properties 351
8.2. Localization 366
8.3. Support of central idempotents 369
8.4. Some module isomorphisms 374

9. Group Cohomology 379

9.1. Preliminaries 379
9.2. The standard resolution 392
9.3. Change of groups 400
9.4. Restriction, corestriction and inflation 403
9.5. Stable submodules 408
9.6. Swan's theorem 411
9.7. The Hochschild-Serre exact sequence 414
9.8. Group extensions and cohomology 420

10. Graded Algebras and Crossed Products **427**

10.1. Group-graded algebras 427
10.2. Crossed products 437
10.3. Simple crossed products 449
10.4. Semilinear monomial representations 451
10.5. Crossed products over simple rings 455
10.6. Equivalent crossed products 461

11. Algebras over Fields **469**

11.1. Splitting fields 469
11.2. Intertwining numbers and semisimplicity 479
11.3. Separable algebras 483
11.4. Blocks, characters and Cartan matrices 487
11.5. The Deuring-Noether theorem 492

12. The Brauer Group **495**

12.1. Central simple algebras 495
12.2. The Brauer group 502
12.3. Tensor product of division algebras 508
12.4. The Brauer group and crossed products 510

13. Indecomposable Modules and Ground Field Extensions **519**

13.1. Preliminary results 519
13.2. Homogeneous components 523
13.3. Idempotent liftings and group actions 526
13.4. Behaviour of indecomposable modules under ground field extension 529

14. The Schur Index **539**

14.1. Preliminary results 539
14.2. Behaviour of simple modules under ground field extensions 543
14.3. The Witt-Fein's theorem 546
14.4. The Schur index 549
14.5. Linear independence of characters 553

15. Frobenius and Symmetric Algebras **555**

15.1. Elementary properties of Frobenius and symmetric algebras 555

15.2. Cogenerators 559
15.3. Quasi-Frobenius algebras 563
15.4. Frobenius algebras 571
15.5. Symmetric algebras 580

16. Dedekind Domains and Discrete Valuation Rings 591

16.1. Integrally closed domains 591
16.2. Dedekind domains 599
16.3. Discrete valuation rings 611
16.4. Completions 613
16.5. Complete discrete valuation rings 615

Bibliography xix

Notation xliii

Index li

Part II Introduction to Group Representations 621

17. Generalities 623

17.1. Definitions and elementary properties 623
17.2. Splitting fields 631
17.3. Counting simple modules over splitting fields 634
17.4. Brauer's permutation lemma 637
17.5. Counting simple modules over arbitrary fields 639
17.6. The socle and Reynolds ideal 643
17.7. Inner and outer tensor products 647
17.8. Representations of direct products 653
17.9. Changing the characteristic 662
17.10. Dimensions of absolutely simple modules 668

18. Induced Modules 671

18.1. Restriction and induction 671
18.2. Induction and semisimplicity 681
18.3. Induction of dual and contragredient modules 683
18.4. Reciprocity theorems 691

18.5. Tensor products 699
18.6. Mackey's theorems 702
18.7. Counting induced modules 709
18.8. The relative trace map 712
18.9. Induction and relative projectivity 716
18.10. An application: Knörr's theorem 722
18.11. Clifford's theorem 725
18.12. Monomial modules 727

Part III Introduction to Characters **731**

19. An Invitation to Characters **733**

19.1. Induced characters 733
19.2. Orthogonality relations 742
19.2.A. Preliminary results 743
19.2.B. Orthogonality relations 744
19.2.C. Intertwining numbers and applications 750
19.3. Class functions and character rings 755
19.3.A. Generalities 755
19.3.B. Splitting fields 758
19.3.C. $\mathbb{C}$-characters 761
19.3.D. Prime and maximal ideals 766
19.4. Representations of abelian groups 769
19.5. Inductive sources 771

20. Induction Theorems and Applications **779**

20.1. The Witt-Berman's induction theorem 779
20.2. Brauer's theorems 786
20.3. Rational valued characters 789

21. Central, Faithful and Permutation Characters **795**

21.1. Central characters 796
21.2. Character kernels and faithful characters 800
21.2.A. General properties 800
21.2.B. Gaschütz's theorem 804
21.2.C. Faithful irreducible characters 808
21.2.D. Extra-special p-groups 811
21.3. Permutation characters 813
21.4. Irreducible characters of p' and p-power degrees 815

22. Character Tables **823**

22.1. Group information 823
22.2. Galois actions 828
22.3. Character tables for A_5 and S_5 832

23. Zeros of Characters **837**

23.1. Burnside's theorem 838
23.2. Characters vanishing off subgroups 840
23.3. Gallagher's theorems 842
23.4. Applications and related results 845
23.5. $\check{Z}$mud's theorems 847

24. Characters, Conjugate Elements and Commutators **855**

24.1. Products of conjugate elements 855
24.2. Characters and commutators 860

25. The Frobenius-Schur Indicator **867**

25.1. Unitary matrices 867
25.2. The Frobenius-Schur indicator 874

26. Characters and Hall subgroups **881**

26.1. An excursion into group theory 881
26.2. Characters and Hall subgroups 894
26.3. Degrees of faithful characters 897

27. Extensions of Characters **907**

27.1. A general criterion 908
27.2. Gallagher's theorems 910
27.3. Two results of Thompson 915
27.4. Thompson's theorems 919
27.5. Character restriction property 921
27.5.A. Introduction 921
27.5.B. Preliminary results 922
27.5.C. Two theorems of Isaacs 926

28. Irreducible Constituents and Conjugacy Classes **931**

28.1. Irreducible constituents of induced characters 931

28.2. Characters and conjugacy classes 935

29. Fixed-Point Spaces and Powers of Characters 939

29.1. Characters and fixed-point spaces 939
29.2. Powers of characters 947

30. Determinants of Characters 953

30.1. Determinants of characters 953
30.2. Character-theoretic transfer 966

31. Tensor Induction of Characters 977

31.1. Tensor product of $n \geqslant 2$ modules 978
31.2. Tensor induced modules 980
31.3. Tensor induced characters 985
31.4. Tensor induced class functions 989
31.5. An application to characters of central products 1000

32. Knörr's Generalized Character 1005

32.1. Definition 1006
32.2. Reduction to $C_G(v)$ 1008
32.3. The abelian case 1013
32.4. The main result 1020

33. Characters of Centralizer Rings 1023

33.1. Characters of $eFGe$ 1023
33.1.A. Characters of centralizer rings 1024
33.1.B. Bases and block idempotents 1027
33.1.C. Applications to degrees of characters 1031
33.2. Characters of $C_{FG}(H)$ 1034

34. Characters and Relative Normal Complements 1041

34.1. Relative normal complements 1042
34.2. π-Sections 1043
34.3. A lifting operator 1045
34.4. Theorems of Dade, Brauer and Suzuki 1051
34.5. Some generalized characters 1055
34.6. Generalized characters and π-sections 1063
34.7. Complements and character extensions 1068

35. Isometries and Generalized Characters **1075**

35.1. π-Induction 1075
35.2. Normal π'-subgroups 1082
35.3. Isometries and generalized characters 1085

36. Exceptional Characters **1095**

36.1. Trivial intersection sets 1095
36.2. Coherent sets of characters 1101
36.2.A. Preliminary results 1101
36.2.B. The main theorem and its applications 1111
36.3. Exceptional subsets 1120
36.4. Special classes 1127

37. Frobenius Groups **1135**

37.1. Preliminary results 1135
37.2. Thompson's criterion for p-nilpotence 1140
37.3. Fixed-point-free automorphisms 1146
37.4. Structure of Frobenius groups 1154
37.5. Characters of Frobenius groups 1159
37.6. Coherence 1164

38. Applications of Characters **1179**

38.1. Burnside's p^aq^b theorem 1179
38.2. Wielandt's theorems 1180
38.3. Generalized quaternion Sylow subgroups 1186
38.4. The Brauer-Suzuki-Wall theorem 1196
38.5. Applications to $U(\mathbb{Z}G)$ 1207
38.5.A. Preliminary results 1208
38.5.B. Torsion units 1211
38.5.C. The isomorphism class of $U(\mathbb{Z}G)$ 1213
38.5.D. Effective construction of units of $\mathbb{Z}G$ 1216
38.5.E. Cyclic groups 1223

Bibliography **1231**

Notation **1255**

Index **1263**