

# Contents

|   |            |
|---|------------|
| <b>Preface</b>  | <b>vii</b> |
| <b>Part I : <math>G</math>-Algebras and Puig's Theory</b> | <b>1</b>   |
| <b>1. Introduction to <math>G</math>-Algebras</b>         | <b>3</b>   |
| 1.1. Definitions and examples                             | 4          |
| 1.2. The relative trace map                               | 9          |
| 1.3. Idempotents in $G$ -algebras                         | 17         |
| 1.4. Skew group rings over $G$ -algebras                  | 21         |
| 1.5. Defect groups in $G$ -algebras                       | 27         |
| 1.5.A. General facts                                      | 28         |
| 1.5.B. Characterizations via Brauer morphism              | 34         |
| 1.5.C. Additional information                             | 35         |
| 1.6. Permutation $G$ -algebras                            | 37         |
| 1.7. The Brauer morphism                                  | 40         |
| 1.7.A. General information                                | 41         |
| 1.7.B. Brauer morphism and permutation $G$ -algebras      | 43         |
| 1.7.C. Brauer pairs                                       | 48         |
| 1.7.D. The group algebra case                             | 54         |
| 1.8. Weak homomorphisms and direct embeddings             | 57         |
| 1.9. Inflated $G$ -algebras                               | 60         |
| 1.10. $G$ -algebras and strongly graded algebras          | 63         |
| 1.11. Trace of radicals                                   | 66         |
| 1.12. Brauer's first main theorem for $G$ -algebras       | 77         |
| 1.12.A. The main theorem                                  | 77         |

|  |            |
|--|------------|
| 1.12.B. Applications to permutation $G$ -algebras        | 81         |
| 1.12.C. Additional results                               | 83         |
| 1.13. Matrix $G$ -algebras                               | 90         |
| 1.14. Applications                                       | 96         |
| 1.14.A. A congruence modulo commutator submodule         | 97         |
| 1.14.B. Some properties of idempotents                   | 100        |
| 1.14.C. Additional information                           | 106        |
| <b>2. Interior <math>G</math>-Algebras</b>               | <b>111</b> |
| 2.1. Definitions and elementary properties               | 112        |
| 2.2. Characterizations of defect groups and applications | 130        |
| 2.3. Restriction and induction of interior $G$ -algebras | 135        |
| 2.4. Subsidiary results                                  | 145        |
| 2.5. Blocks, vertices and interior $G$ -algebras         | 148        |
| 2.6. Inflated interior $G$ -algebras                     | 152        |
| 2.7. Skew group rings over interior $G$ -algebras        | 154        |
| 2.8. Morita contexts and direct embeddings               | 165        |
| 2.9. Source algebras                                     | 172        |
| <b>3. Puig's Theory : Part A</b>                         | <b>181</b> |
| 3.1. Points in semiperfect rings                         | 182        |
| 3.2. Elementary properties of pointed groups             | 192        |
| 3.2.A. General facts                                     | 192        |
| 3.2.B. Specialization to $End_R(V)$                      | 207        |
| 3.3. Simple $G$ -algebras and multiplicity modules       | 211        |
| 3.3.A. Preliminaries                                     | 211        |
| 3.3.B. Multiplicity modules for simple $G$ -algebras     | 215        |
| 3.3.C. Multiplicity modules for pointed groups           | 222        |
| 3.4. Defect pointed groups                               | 225        |
| 3.4.A. Introduction                                      | 225        |
| 3.4.B. General theory                                    | 226        |
| 3.4.C. Applications to $End_R(V)$                        | 238        |
| <b>4. Puig's Theory : Part B</b>                         | <b>241</b> |
| 4.1. Puig correspondence                                 | 242        |

|           |   |            |
|-----------|---|------------|
| 4.2.      | Multiplicity modules and defect pointed groups        | 252        |
| 4.2.A.    | General facts   | 252        |
| 4.2.B.    | Simple multiplicity modules                           | 255        |
| 4.2.C.    | Free multiplicity modules                             | 260        |
| 4.2.D.    | Barker's theorem                                      | 262        |
| 4.3.      | Green correspondence for pointed groups               | 266        |
| 4.4.      | An internal characterization of source algebras       | 270        |
| 4.5.      | Restrictions, exomorphisms and direct embeddings      | 272        |
| 4.6.      | Direct embeddings and local control                   | 277        |
| <b>5.</b> | <b>Puig's Theory : Part C</b>                         | <b>291</b> |
| 5.1.      | Pointed groups on interior $G$ -algebras              | 292        |
| 5.2.      | Pointed groups and ideals                             | 300        |
| 5.3.      | Multiplicity algebras of maximal local pointed groups | 308        |
| 5.4.      | Applications to trace of radicals                     | 312        |
| 5.5.      | Points of $R$ -simple interior $G$ -algebras          | 321        |
| 5.6.      | Pointed groups and Brauer pairs                       | 324        |
| 5.7.      | Preparatory results for the group algebra case        | 326        |
| 5.8.      | Pointed groups on group algebras                      | 332        |
| 5.8.A.    | General facts   | 332        |
| 5.8.B.    | Point correspondences                                 | 337        |
| 5.8.C.    | Some additional results                               | 345        |
| <b>6.</b> | <b>Bilinear Forms on <math>G</math>-Algebras</b>      | <b>353</b> |
| 6.1.      | Preliminary results                                   | 354        |
| 6.1.A.    | General facts   | 354        |
| 6.1.B.    | The group algebra case                                | 359        |
| 6.1.C.    | Comparison with a bilinear form of Green              | 363        |
| 6.2.      | The Broué - Robinson's theorem                        | 367        |
| 6.3.      | Applications to the group algebra                     | 373        |

|  |            |
|--|------------|
| <b>Part II : Block Theory</b>  | <b>379</b> |
| <b>7. Preliminaries</b>  | <b>381</b> |
| 7.1. Blocks and their defect groups                                  | 381        |
| 7.1.A. Block decompositions  | 382        |
| 7.1.B. Defect groups and their characterizations                     | 384        |
| 7.1.C. Defect groups and normal $p$ -subgroups                       | 388        |
| 7.1.D. Induced block decompositions                                  | 390        |
| 7.1.E. Blocks of central separable group algebras                    | 392        |
| 7.1.F. Changing the characteristic                                   | 393        |
| 7.1.G. Field coefficients  | 394        |
| 7.2. Defect groups are Sylow intersections                           | 398        |
| 7.3. Nagao's theorem   | 401        |
| 7.4. Defect groups and vertices                                      | 405        |
| 7.5. Blocks of defect zero   | 410        |
| 7.6. The Jacobson radical of $Z(FG)$                                 | 417        |
| 7.7. Some properties of idempotents                                  | 420        |
| 7.7.A. Elementary technical lemmas                                   | 421        |
| 7.7.B. Theorems of Iizuka, Watanabe and Osima                        | 426        |
| 7.7.C. An application  | 429        |
| 7.8. The principal block   | 431        |
| 7.9. The principal block idempotent                                  | 438        |
| 7.10. Kernels of blocks  | 444        |
| 7.11. Exponents of defect groups of blocks                           | 451        |
| <b>8. Block Inductions and Brauer's Theorems</b>                     | <b>457</b> |
| 8.1. Brauer's first main theorem for semilocal coefficient rings     | 458        |
| 8.2. Central characters and block inductions                         | 462        |
| 8.2.A. Central characters and blocks                                 | 462        |
| 8.2.B. Brauer's block induction                                      | 466        |
| 8.2.C. Extended block induction                                      | 482        |
| 8.2.D. Central characters and block inductions in the semilocal case | 488        |
| 8.3. Theorems of Conlon, Green, Nagao and Watanabe                   | 497        |
| 8.4. Brauer's second main theorem                                    | 509        |

|            |   |            |
|------------|---|------------|
| 8.5.       | Brauer's third main theorem                             | 511        |
| <b>9.</b>  | <b>Counting Blocks With a Given Defect Group</b>        | <b>515</b> |
| 9.1.       | Preliminary results                                     | 515        |
| 9.2.       | Robinson's theorem                                      | 520        |
| 9.3.       | The $F$ -dimension of $J(Z(FG))$                        | 523        |
| 9.4.       | Existence of $p$ -blocks with a given defect group      | 526        |
| <b>10.</b> | <b>Blocks and Normal Subgroups</b>                      | <b>535</b> |
| 10.1.      | Preliminaries   | 536        |
| 10.2.      | Block covers  | 545        |
| 10.3.      | Block covers and central characters                     | 555        |
| 10.4.      | Extended and Brauer roots                               | 560        |
| 10.5.      | Block covers and defect groups                          | 564        |
| 10.6.      | Brauer's extended first main theorem                    | 573        |
| 10.7.      | Defect groups and normal subgroups                      | 576        |
| 10.8.      | Brauer correspondence for covering blocks               | 580        |
| 10.9.      | Regular and weakly regular blocks                       | 586        |
| 10.10.     | The Fong correspondence                                 | 591        |
|            | 10.10.A. Preliminary results                            | 591        |
|            | 10.10.B. The Fong correspondence                        | 596        |
| 10.11.     | A decomposition theorem and applications                | 601        |
|            | 10.11.A. Preliminaries                                  | 601        |
|            | 10.11.B. A decomposition theorem                        | 607        |
|            | 10.11.C. Groups with primary and quasi-primary blocks   | 612        |
| 10.12.     | Blocks of groups and factor groups                      | 616        |
|            | 10.12.A. Preliminary results                            | 616        |
|            | 10.12.B. Main theorems                                  | 620        |
| <b>11.</b> | <b>Blocks With Normal Defect Groups</b>                 | <b>627</b> |
| 11.1.      | Inertial subalgebras                                    | 628        |
|            | 11.1.A. Preliminaries                                   | 628        |
|            | 11.1.B. Existence and conjugacy of inertial subalgebras | 634        |
| 11.2.      | Blocks of crossed products                              | 643        |
|            | 11.2.A. Preliminaries                                   | 643        |

|   |            |
|---|------------|
| 11.2.B. Main theorems                                     | 649        |
| 11.3. Structure of blocks with normal defect groups       | 660        |
| <b>12. Blocks and Characters : Part A</b>                 | <b>671</b> |
| 12.1. Some general facts                                  | 672        |
| 12.1.A. General background                                | 672        |
| 12.1.B. Central endomorphisms                             | 677        |
| 12.1.C. Partitions of bases                               | 680        |
| 12.2. $D$ -blocks   | 682        |
| 12.3. Blocks of characters : general coefficient fields   | 693        |
| 12.4. Blocks of characters : splitting coefficient fields | 703        |
| 12.5. Two applications of block orthogonality             | 717        |
| 12.6. Central characters corresponding to induced blocks  | 719        |
| 12.7. Characters and extended block induction             | 723        |
| 12.8. Characters of height zero                           | 726        |
| 12.9. Blocks, characters and $\alpha$ -covering groups    | 727        |
| 12.9.A. Generalities                                      | 727        |
| 12.9.B. An application                                    | 737        |
| 12.10. Characters and blocks with normal defect groups    | 738        |
| <b>13. Blocks and Characters : Part B</b>                 | <b>743</b> |
| 13.1. Characterizations of defects of blocks              | 744        |
| 13.2. Recognizing characters in the same block            | 751        |
| 13.2.A. General coefficient fields                        | 751        |
| 13.2.B. Splitting coefficient fields                      | 753        |
| 13.3. Some open problems                                  | 759        |
| 13.4. An integral matrix                                  | 763        |
| 13.4.A. Introduction                                      | 763        |
| 13.4.B. An integral matrix                                | 764        |
| 13.5. Counting irreducible characters in blocks           | 772        |
| 13.5.A. The Brauer - Feit theorem                         | 772        |
| 13.5.B. Inequalities for block-theoretic invariants       | 775        |
| 13.5.C. Characterizations of the case $k(B) = 2$          | 777        |
| 13.5.D. Situations in which $k(B)$ is known precisely     | 780        |
| 13.5.E. Central separable blocks                          | 784        |
| 13.6. Generalized decomposition numbers                   | 792        |

|  |            |
|--|------------|
| 13.7. Heights of $R$ -generalized characters             | 807        |
| <b>14. Blocks and Characters : Part C</b>                | <b>815</b> |
| 14.1. Block partitions, $p$ -sections and multiplicities | 816        |
| 14.1.A. Block partititons of conjugacy classes           | 816        |
| 14.1.B. Multiplicities                                   | 819        |
| 14.1.C. The trivial $p$ -section                         | 827        |
| 14.1.D. Lower defect groups                              | 833        |
| 14.2. Multiplicities of defect groups                    | 834        |
| 14.2.A. Preliminary results                              | 834        |
| 14.2.B. The main result                                  | 838        |
| 14.2.C. Applications                                     | 842        |
| 14.3. Subsections  | 844        |
| 14.4. Characters, normal subgroups and block covers      | 848        |
| 14.5. Subpairs and Brauer nets                           | 858        |
| 14.5.A. Generalities                                     | 858        |
| 14.5.B. Centralizer, normalizer and extremal subpairs    | 866        |
| 14.5.C. Major subpairs                                   | 870        |
| 14.5.D. Brauer nets                                      | 872        |
| 14.6. Vertices of virtually irreducible lattices         | 876        |
| 14.7. Blocks, characters and subgroups                   | 879        |
| 14.7.A. General results                                  | 879        |
| 14.7.B. Characters and block induction                   | 884        |
| <b>Bibliography</b>                                      | <b>889</b> |
| <b>Notation</b>  | <b>953</b> |
| <b>Index</b>   | <b>961</b> |