

Table of contents

Introduction	13
Chapter I. Algebraic background	23
Section 1. On exponential functors	23
Definition 1.1. Multiplicative category, exponential functor and polynomial algebra, Hopf algebra – Definition 1.2. Subadditive and submultiplicative functors, compatible natural transformations – Lemma 1.3. E_2SA and TE_1A are algebras when E_i is exponential, S subadditive, T submultiplicative – Lemma 1.4. φ is a morphism of algebras – Lemma 1.5. About coalgebras – Proposition 1.6. $E \text{ Hom}_R(-, M) \rightarrow \text{Hom}_Z(E-, M)$ is a morphism of graded algebras for $E = \wedge, P$ – Lemmas 1.10, 1.11. The structure of $\text{Hom}(PZ^n, Z)$ – Lemmas 1.12, 1.13. More about $\text{Hom}(E-, M)$ – Proposition 1.14. The structure of $\text{Hom}(EA, M)$, $E = \wedge, P$ – Definition 1.15. Polynomial algebras with divided powers – Proposition 1.16. $E \text{ Hom}(-, M) \rightarrow \text{Hom}(E-, M)$ for $E = P \otimes \wedge$ – Lemma 1.17. About the natural map η_A – Proposition 1.18. The coalgebra $\text{Hom}(G, Z)$ – Corollary 1.19. The duality of polynomial algebras and algebras with divided powers – Theorem 1.22. The map $P_R A \otimes_R \wedge_R B \rightarrow \text{Hom}(PA \otimes \wedge B, R)$ – Proposition 1.23. about $\text{Hom}_S(K \otimes L, A) \rightarrow \text{Hom}_S(L, \text{Hom}(K, A))$ for complexes	
Section 2. The arithmetic of certain spectral algebras	43
Definition 2.1. Spectral algebra, edge algebra – Lemmas 2.2, 2.3. The derivations d, d_φ – Definition 2.4. The functors E_2, E_3 – Lemma 2.5. The cohomology map preserves multiplication – Lemma 2.6. Definition of the cohomology map ψ – Definition 2.7. The first edge algebra and	

$B^{2p}(\varphi)$ – Definition 2.8. Integral elements in rings, weakly principal ideal rings – Definition 2.10. The formalism of the derivation d_φ on $E_2(\varphi)$ – Definition 2.11. The elementary morphisms – Proposition 2.12. The structure of the edge terms in $E_3(\varphi)$ – Lemma 2.13. The elements of $\ker d_\varphi$ – Lemma 2.14. The elements of $\text{im } d_\varphi$ – Proposition 2.16. $u \otimes a'_s \rightarrow ua'_s : E_3^{\text{II}}(\varphi) \otimes a'_s \rightarrow E_3(\varphi)$ is injective – Proposition 2.17. The terms next to the edge terms – An explicit example – Corollary 2.18. The terms next to the edge terms for a principal ideal domain as coefficient ring – Lemma 2.20. Passage to the ring of quotients in the coefficient ring – Proposition 2.21. $E_2(\varphi \otimes \psi) \cong E_2(\varphi) \otimes E_2(\psi)$ – Proposition 2.24, 2.25. Conditions under which d_φ is exact – Proposition 2.26. The exactness of d_φ within the ground ring extension – Lemma 2.31. Elementary morphisms yielding the same E_3 – Proposition 2.32. The case that φ is a homothety – Proposition 2.33. Elementary morphisms which differ by a scalar – Proposition 2.34. $E_3(\varphi_1 \otimes \varphi_2) \cong E_3(\varphi_2)$ if $\text{im } d(\varphi_1)$ is flat – Proposition 2.35. An inductive process to compute $E_3(\varphi)$ if the ground ring is a principal ideal domain – Theorem I. $E_3(\varphi)$ is generated as a $(P \text{ coker } \varphi)$ -module by M – Definition 2.39. Definition of λ and $E_2^*(\varphi)$ – Lemma 2.40. The differential modules $(E_2(\varphi), d')$ – Proposition 2.42. About the structure of $E_3(\varphi)$ – Propositions 2.43, 2.44. About the PA -module structure of $E_r(\varphi)$ – Propositions 2.47, 2.48. Non-injective elementary morphisms.

Section 3. Some analogues of the results about spectral algebras with dual derivations

74

Lemma 3.1. The differential and derivative ∂_φ – Definition 3.2, 3.3. The spectral algebras $E_r[\varphi]$, $E_r\{\varphi\}$ – Lemma 3.4. $E_2[-]$ is an exponential functor – Lemma 3.5. About $\partial d + d\partial$ – Proposition 3.6. The edge algebra $E_3^{\text{II}}[\varphi]$ – Definition 3.6a. R -coalgebras, differential graded coalgebras, differential graded Hopf algebras – Proposition 3.7. $E_2\{\varphi\}$ is a differential bi-graded Hopf algebra relative to d_φ and ∂_φ – Lemma 3.8. The cofunctor $f \rightarrow E_2\{\text{Hom}_s(f, R)\}$ – Lemma 3.9. About the structure of finite abelian groups – Definition 3.10. Standard resolution of a finite abelian group – Lemma 3.11. The uniqueness of standard resolutions – Lemma 3.12. The four term exact sequence derived from an injection – Lemma 3.13. Isomorphic version of $\ker \wedge^p \text{Hom}(f, A)$ – Proposition 3.14. The edge terms in $E_3(\text{Hom}(f, R))$ – Corollary 3.15. The morphism $P_R \text{Ext}(G, R) \otimes_R \text{Hom}(\wedge G, R) \rightarrow E_3(\text{Hom}(f, R))$ – Corollary 3.16. The functoriality of this morphism – Propositions 3.17, 3.18. The isomorphisms $H(R/Z \otimes E_2(f)) \rightarrow E_3(f) \rightarrow H(E_2(f)^\wedge)$.

Section 4. The Bockstein formalism

85

Lemmas 4.1, 4.2, 4.3, 4.4. Some diagram chasing – Definition 4.5. The

definition of pre-Bockstein diagrams and standard Bockstein diagrams – Lemmas 4.6, 4.7, 4.8. About the Bockstein formalism – Proposition 4.9. An isomorphism of exact sequences – Lemma 4.10. More diagram chasing – Proposition 4.11. Sufficient conditions for the Bockstein formalism for complexes – Proposition 4.12. When is the Bockstein differential a derivation? – Corollaries 4.13, 4.14. The standard situation – Proposition 4.15. The Bockstein formalism for the cohomology of groups and complexes – Proposition 4.16. The Bockstein formalism for the spectral algebras $\mathcal{E}_2(\varphi)$ of Section 2 – Corollary 4.17. A particular case of 4.16.	
Chapter II. The cohomology of finite abelian groups	98
Section 1. Products	98
Definition 1.1. The construction of κ – Definition 1.2. The construction of λ – Lemma 1.3. Tensoring resolutions – Corollary 1.4 – Lemma 1.5. The Künneth theorem – Theorem 1.6. The resolution of augmented Hopf algebras – Theorem II. Cohomology and the tensor product of Hopf algebras – Corollary 1.7. About $H(G_1 \times G_2, R)$ – Corollary 1.8. A Künneth theorem for $H(G_1 \times G_2, R)$ – Corollary 1.9. A special case of 1.8 – Corollary 1.10. $H(G_1 \times G_2, R)$ for cyclic G_1 – Corollary 1.11. $H(G_1, R) \otimes \cdots \otimes H(G_n, R) \cong H(G_1 \times \cdots \times G_n, R)$ – Corollary 1.12. About the annihilator of $H^+(G_1 \times G_2, R)$ – Corollary 1.13. About the exponent of $H^+(G_1 \times G_2, R)$ – Corollary 1.14. The exponent of $H^+(G, M)$ for a finite abelian group G and arbitrary M – Corollary 1.15. $H(G, M) \otimes N \cong H(G, M \otimes N)$.	
Section 2. Special free resolutions for finite abelian groups	113
Definition 2.1. Special elements in the group ring of a finite abelian group – Lemma 2.2. About $\partial: \wedge \rightarrow \wedge A^+$ – Lemma 2.3. $d\partial + \partial d = 0$ – Lemma 2.4. The coderivation $D = d + \partial$ – Definition 2.5. $\mathcal{E}(f)$ and $\hat{\mathcal{E}}(f)$ – Lemma 2.6. $\hat{\mathcal{E}}$ is exponential – Lemma 2.7. $\hat{\mathcal{E}}(f)$ exact – special case – Lemma 2.8. $0 \leftarrow \mathbf{Z} \leftarrow \hat{\mathcal{E}}(f)$ is a resolution – Lemma 2.9. $R \otimes_S A \rightarrow \text{Hom}_S(\text{Hom}_S(A, S), R)$ – Theorem III. Fundamental theorem about the cohomology of finite abelian groups – Lemma 2.10. $H^i(G, \mathbf{R}/\mathbf{Z}) \cong H^{i+1}(G, \mathbf{Z})$ – Proposition 2.11. Various isomorphisms involving $H(G, \mathbf{R}/\mathbf{Z})$ – Lemma 2.12. A categorical lemma – Theorem 2.13. The morphism $\omega: P_R \text{Ext}(G, R) \otimes_R \text{Hom}(\wedge G, R) \rightarrow H(G, R)$ – Lemma 2.14. A lemma involving the bar resolution – Proposition and Corollaries 2.15–2.18. A relation between the bar resolution and the bi-resolution.	
Section 3. About the cohomology of finite abelian groups in the case of trivial action	128
Definition 3.1. Recapitulation of the standard resolution – Lemma 3.2.	

A group theoretical lemma — Proposition 3.3. A special case of Theorem 2.13 — Definition 3.4. The z -constituent of a group — Proposition 3.5. Splitting the z -constituent in $H(G, R)$ — Theorem 3.6. The cohomology of $\mathbf{Z}(z)^n$ — Corollary 3.7. The Poincaré series for Theorem 3.6 — Corollary 3.8. The additive structure of $H(G, \mathbf{Z}(z))$ for an arbitrary G whose exponent divides z — Proposition 3.9. Decomposing $H(G, \mathbf{Z})$ — Theorem 3.11. A structure theorem for $H(G, \mathbf{Z})$ — Theorem IV. About the structure of the ring $H(G, \mathbf{Z})$ — Corollaries 3.13–3.15. A minimal generating module for $H(G, \mathbf{Z})$ — Theorem V. The complete structure of $H(G, R)$ if R is a field — Example — Proposition 3.16. $H(G_1 \times G_2, R)$ for groups G_1, G_2 with relatively prime order — Propositions 3.17, 3.18. About the Bocksteins in low dimension — Proposition 3.19. A global version of the previous results.

Section 4. Appendix to Section 3: The low dimensions 148

Proposition 4.1. A list for $H^i(G, R)$ for $i < 6$ — Proposition 4.2. $(\wedge G)^\wedge \cong \wedge \hat{G}$ — Proposition 4.3. A list for $H^i(G, R/\mathbf{Z})$ for $i < 4$ — A remark about Schur's multiplier — Two dimensional cohomology and central extensions.

Chapter III. The cohomology of classifying spaces of compact groups . . . 155

Section 1. The functor h 155

Definition 1.1. The join of two spaces, the iterated join — Proposition 1.2. A Künneth theorem for the join and the relation with the standard Künneth theorem — Corollaries 1.3, 1.4. The acyclicity of iterated joins — Definition 1.5. The spectrum of universal spaces for G and the spectrum of classifying spaces for G . Classifying spaces up to n — Lemma 1.6. The existence of spectra of universal spaces — Definition 1.7. The Milnor spectrum of universal spaces (resp. classifying spaces) for G — Proposition 1.8. Properties of the Milnor spectrum — Definition 1.9. The definition of the functor h — Proposition 1.10. The independence of h from the choice of universal spaces — Proposition 1.11, 1.12. h transforms projective limits into direct limits — Proposition 1.13, 1.14. The Künneth theorem for h — Corollaries 1.15, 1.16. Comments on $h(G' \times G, R)$ — Propositions 1.17–1.20. The Bockstein formalism for h .

Section 2. The functor h for finite groups 166

Definition 2.1. Simplicial objects — Proposition 2.2. Products, equalizers, etc. for simplicial sets — Definition 2.3. Group actions on simplicial sets — Lemma 2.4, 2.5. Free simplicial modules — Proposition 2.6. The equivalence of $h(G, R)$ with $H(G, R)$ for finite G — Corollary 2.7. The computation of $h(G, R)$ for totally disconnected compact G .

Chapter IV. Kan extensions of functors on dense categories	172
Section 1. Dense categories and continuous functors	173
Lemma 1.1. The functor LIM — Lemma 1.2. The functor $S^{\mathfrak{D}}$ — Definition 1.3. \mathfrak{D} -continuous functors — Definition 1.4. The comma category — Example 1.5. The category of Lie groups is dense in the category of compact groups — Lemma 1.6. A uniqueness statement for natural transformations — Definition 1.7. Dense subcategories — Example 1.8. Continuation of Example 1.5 — Definition 1.9. Extendable functors — Definition 1.10. Compatible functors — Definition 1.11. Strictly dense subcategories — Proposition 1.12. Extending extendable functors — Definition 1.13. Kan extensions — Theorem 1.14. The Kan extension existence theorem — Theorem 1.15. Density theorem for the category of compact groups.	
Section 2. Multiplicative Hopf extensions	187
Definition 2.1. Freely generated categories — Theorem 2.2. The existence and uniqueness of Hopf extensions — Corollary 2.3. Hopf extensions of functors on compact abelian Lie groups — Corollary 2.4. Hopf extensions of functors on compact connected Lie groups — Corollary 2.5. A uniqueness theorem for exponential functors on compact abelian groups — Proposition 2.6. The exterior algebra functor for compact abelian groups — Lemma 2.7. The properties of the exterior algebra functor — Lemma 2.8. About the functor $\text{Hom}(\wedge -, R)$ — Lemma 2.9. The dual of the exterior algebra of a compact abelian group — Lemma 2.10. $R \otimes \text{Hom}(G, K) \cong \text{Hom}(G, R)$.	
Chapter V. The cohomological structure of compact abelian groups	203
Section 1. The cohomologies of connected compact abelian groups	203
Lemmas 1.1.—1.6. Continuous exponential functors on compact connected abelian groups — Lemmas 1.7, 1.8. Change of coefficients — Theorem 1.9. The structure theorem for cohomology theories on compact connected abelian groups — Theorem 1.10. The singular cohomology on compact connected abelian groups — Corollary 1.11. The algebraic cohomology of a finitely generated abelian group.	
Section 2. The space cohomology of arbitrary compact abelian groups	210
Theorem VI. The structure theorem for topological cohomology.	
Section 3. The canonical embedding of \hat{G} in kG	212
Theorem 3.1. $\hat{G} = h^2(G, \mathbb{Z})$.	
Section 4. Cohomology theories for compact groups over fields as coefficient domains	215

Lemmas 4.1, 4.2. Exponential functors on compact abelian groups — Theorem VII. The algebraic cohomology of a compact abelian group over a field — Corollary 4.3. The algebraic cohomology of a compact abelian group with real coefficients — Theorem 4.4. The algebraic cohomology over a finite prime field and the Bockstein differential.	
Section 5. The structure of h for arbitrary compact abelian groups and integral coefficients	219
Proposition 5.1. Splitting a connected group — Proposition 5.2. The cohomology of compact abelian Lie groups — Propositions 5.3, 5.4. The maps induced in cohomology by the inclusion of the connected identity component and its cokernel — Theorem VIII. The principal theorem for integral cohomology — Lemma 5.5. Reducing an abelian group — Lemma 5.6. The cohomology of a p -adic group — Proposition 5.7. Classification of compact abelian groups with compact classifying space.	
Chapter VI. Appendix. Another construction of the functor h (by Eric C. Nummela)	224
Proposition 1. About the graph of \langle for a topological monoid acting on a space — Proposition 2. Properties of the Dold-Lashof spectrum.	
Bibliography	229
List of notations	232
Index	235