

# Contents

## Chapter 6

### Spherical Functions — The General Theory

<b>Introduction</b> . . . . .	1
<b>6.1 Fundamentals</b> . . . . .	2
6.1.1 Spherical Functions – Functional Properties . . . . .	2
6.1.2 Spherical Functions – Differential Properties . . . . .	16
<b>6.2 Examples</b> . . . . .	20
6.2.1 Spherical Functions on Motion Groups . . . . .	20
6.2.2 Spherical Functions on Semi-Simple Lie Groups . . . . .	30

## Chapter 7

### Topology on the Dual Plancherel Measure

<b>Introduction</b> . . . . .	44
<b>7.1 Topology on the Dual</b> . . . . .	44
7.1.1 Generalities . . . . .	44
7.1.2 Applications to Semi-Simple Lie Groups . . . . .	49
<b>7.2 Plancherel Measure</b> . . . . .	52
7.2.1 Generalities . . . . .	52
7.2.2 The Plancherel Theorem for Complex Connected Semi-Simple Lie Groups . . . . .	54

## Chapter 8

### Analysis on a Semi-Simple Lie Group

<b>Introduction</b> . . . . .	58
<b>8.1 Preliminaries</b> . . . . .	59
8.1.1 Acceptable Groups . . . . .	59
8.1.2 Normalization of Invariant Measures . . . . .	63
8.1.3 Integration Formulas . . . . .	67

8.1.4	A Theorem of Compacity . . . . .	74
8.1.5	The Standard Semi-Norm on a Semi-Simple Lie Group . . . . .	78
8.1.6	Completely Invariant Sets . . . . .	80
<b>8.2</b>	<b>Differential Operators on Reductive Lie Groups and Algebras . . . . .</b>	<b>83</b>
8.2.1	Radial Components of Differential Operators on a Manifold . . . . .	83
8.2.2	Radial Components of Polynomial Differential Operators on a Reductive Lie Algebra . . . . .	93
8.2.3	Radial Components of Left Invariant Differential Operators on a Reductive Lie Group . . . . .	103
8.2.4	The Connection between Differential Operators in the Algebra and on the Group . . . . .	112
<b>8.3</b>	<b>Central Eigendistributions on Reductive Lie Algebras and Groups . . . . .</b>	<b>115</b>
8.3.1	The Main Theorem in the Algebra . . . . .	115
8.3.2	Properties of $F_T - I$ . . . . .	122
8.3.3	The Main Theorem on the Group . . . . .	132
8.3.4	Properties of $F_T - II$ . . . . .	139
8.3.5	Rapidly Decreasing Functions on a Euclidean Space . . . . .	144
8.3.6	Tempered Distributions on a Reductive Lie Algebra . . . . .	149
8.3.7	Rapidly Decreasing Functions on a Reductive Lie Group . . . . .	152
8.3.8	Tempered Distributions on a Reductive Lie Group . . . . .	166
8.3.9	Tools for Harmonic Analysis on $G$ . . . . .	175
<b>8.4</b>	<b>The Invariant Integral on a Reductive Lie Algebra. . . . .</b>	<b>178</b>
8.4.1	The Invariant Integral – Definition and Properties . . . . .	178
8.4.2	Computations in $\mathfrak{sl}(2, \mathbf{R})$ . . . . .	182
8.4.3	Continuity of the Map $f \mapsto \phi_f$ . . . . .	190
8.4.4	Extension Problems . . . . .	202
8.4.5	The Main Theorem . . . . .	211
<b>8.5</b>	<b>The Invariant Integral on a Reductive Lie Group . . . . .</b>	<b>226</b>
8.5.1	The Invariant Integral – Definition and Properties . . . . .	226
8.5.2	The Inequalities of Descent . . . . .	236
8.5.3	The Transformations of Descent . . . . .	242
8.5.4	The Invariant Integral and the Transformations of Descent . . . . .	245
8.5.5	Estimation of $\Phi_f$ and its Derivatives . . . . .	247
8.5.6	An Important Inequality . . . . .	255
8.5.7	Convergence of Certain Integrals . . . . .	258
8.5.8	Continuity of the Map $f \mapsto \Phi_f$ . . . . .	261

## Chapter 9

### Spherical Functions on a Semi-Simple Lie Group

<b>Introduction . . . . .</b>	<b>264</b>
<b>9.1 Asymptotic Behavior of <math>\mu</math>-Spherical Functions on a Semi-Simple Lie Group</b>	<b>265</b>
9.1.1 The Main Results . . . . .	265
9.1.2 Analysis in the Universal Enveloping Algebra . . . . .	269

9.1.3	The Space $\Xi(\mu, \nu)$ . . . . .	281
9.1.4	The Rational Functions $F_\lambda$ . . . . .	286
9.1.5	The Expansion of $\mu$ -Spherical Functions . . . . .	300
9.1.6	Investigation of the $c$ -Function . . . . .	317
9.1.7	Applications to Zonal Spherical Functions . . . . .	325
<b>9.2</b>	<b>Zonal Spherical Functions on a Semi-Simple Lie Group</b> . . . . .	<b>335</b>
9.2.1	Statement of Results – Immediate Applications . . . . .	335
9.2.2	The Plancherel Theorem for $L^2(G)$ . . . . .	344
9.2.3	The Paley-Wiener Theorem for $L^2(G)$ . . . . .	353
9.2.4	Harmonic Analysis in $L^1(G)$ . . . . .	359
<b>9.3</b>	<b>Spherical Functions and Differential Equations</b> . . . . .	<b>367</b>
9.3.1	The Weak Inequality and Some of its Implications . . . . .	367
9.3.2	Existence and Uniqueness of the Indices I . . . . .	372
9.3.3	Existence and Uniqueness of the Indices II . . . . .	377

Chapter 10

**The Discrete Series for a Semi-Simple Lie Group —  
Existence and Exhaustion**

<b>Introduction</b> . . . . .	<b>389</b>
<b>10.1 The Role of the Distributions <math>\Theta_\tau</math> in the Harmonic Analysis on <math>G</math></b> . . . . .	<b>390</b>
10.1.1 Existence and Uniqueness of the $\Theta_\tau$ . . . . .	390
10.1.2 Expansion of $\mathfrak{J}$ -Finite Functions in $\mathcal{C}(G)$ . . . . .	397
<b>10.2 Theory of the Discrete Series</b> . . . . .	<b>400</b>
10.2.1 Existence of the Discrete Series . . . . .	400
10.2.2 The Characters of the Discrete Series I – Implication of the Orthogonality Relations . . . . .	401
10.2.3 The Characters of the Discrete Series II – Application of the Differential Equations . . . . .	404
10.2.4 The Theorem of Harish-Chandra . . . . .	407
<b>Epilogue</b> . . . . .	<b>414</b>

**Appendix**

<b>3 Some Results on Differential Equations.</b> . . . . .	<b>426</b>
3.1 The Main Theorems . . . . .	426
3.2 Lemmas from Analysis . . . . .	428
3.3 Analytic Continuation of Solutions . . . . .	430
3.4 Decent Convergence . . . . .	432
3.5 Normal Sequences of $E$ -Polynomials . . . . .	433
<b>General Notational Conventions</b> . . . . .	<b>450</b>
<b>List of Notations</b> . . . . .	<b>452</b>
<b>Guide to the Literature</b> . . . . .	<b>456</b>
<b>Bibliography</b> . . . . .	<b>460</b>
<b>Subject Index to Volumes I and II</b> . . . . .	<b>484</b>