CONTENTS

PREFACE xi

CHAPTER I. THE THREE-DIMENSIONAL ROTATION GROUP AND THE LORENTZ GROUP 1

1. THE THREE-DIMENSIONAL ROTATION GROUP 1
   1. General definition of a group. 1
   2. Definition of the three-dimensional rotation group. 2
   3. Description of rotations by means of orthogonal matrices. 2
   4. Eulerian angles. 5
   5. The description of rotation by means of unitary matrices. 7
   6. The invariant integral over the rotation group. 13
   7. The invariant integral on the unitary group. 17
2. THE LORENTZ GROUP 18
   1. The general Lorentz group. 18
   2. The complete Lorentz group and the proper Lorentz group. 23

CHAPTER II. THE REPRESENTATIONS OF THE THREE-DIMENSIONAL ROTATION GROUP 25

3. THE BASIC CONCEPTS OF THE THEORY OF FINITE-DIMENSIONAL REPRESENTATIONS 25
   1. Linear spaces. 25
   2. Linear operators. 27
   3. Definition of a finite-dimensional representation of a group. 28
   4. Continuous finite-dimensional representations of the three-dimensional rotation group. 29
   5. Unitary representations. 30
4. IRREDUCIBLE REPRESENTATIONS OF THE THREE-DIMENSIONAL ROTATION GROUP IN INFINITESIMAL FORM 31
   1. Differentiability of representations of the group G_0. 31
   2. Basic infinitesimal matrices of the group G_0. 33
   3. Basic infinitesimal operators of a representation of the group G_0. 35
   4. Relations between the basic infinitesimal operators of a representation of the group G_0. 39
   5. The condition for a representation to be unitary. 41
   6. General form of the basic infinitesimal operators of the irreducible representations of the group G_0. 43

5. THE REALIZATION OF FINITE-DIMENSIONAL IRREDUCIBLE REPRESENTATIONS OF THE THREE-RIMENSIONAL ROTATION GROUP 50
1. The connection between the representations of the group G_0 and the representations of the unitary group $\mathfrak{U}$. 50
2. Spinor representations of the group $\mathfrak{U}$. 51
3. Realization of the representations $\mathfrak{S}_m$ in a space of polynomials. 54
4. Basic infinitesimal operators of the representation $\mathfrak{S}_m$. 56
5. Orthogonality relations. 60
6. THE DECOMPOSITION OF A GIVEN REPRESENTATION OF THE THREE-DIMENSIONAL ROTATION GROUP INTO IRREDUCIBLE REPRESENTATIONS 63
1. The case of a finite-dimensional unitary representation. 63
2. The theorem of completeness. 66
3. General definition of a representation. 68
4. Continuous representations. 70
5. The integrals of vector and operator functions. 73
6. Decomposition of a representation of the group $\mathfrak{U}$ into irreducible representations. 77
7. The case of a unitary representation. 83
CHAPTER III. IRREDUCIBLE LINEAR REPRESENTATIONS OF THE PROPER AND COMPLETE LORENTZ GROUPS 89
7. THE INIFINITESIMAL OPERATORS OF A LINEAR REPRESENTATION OF THE PROPER LORENTZ GROUP 89
1. The infinitesimal Lorentz matrices. 89
2. Relations between the infinitesimal Lorentz matrices. 96
3. The infinitesimal operators of a representation of the proper Lorentz group. 96
4. Relations between the basic infinitesimal operators of a representation. 101
8. DETERMINATION OF THE INFINITESIMAL OPERATORS OF A REPRESENTATION OF THE GROUP $\mathfrak{G}_+$. 103
1. Statement of the problem. 103
2. Determination of the operators H_+, H_-, H_3. 104
3. Determination of the operators F_+, F_-, F_3. 106
4. The conditions of being unitary. 117
9. THE FINITE-DIMESIONAL REPRESENTATIONS OF THE PROPER LORENTZ GROUP 120
1. The spinor description of the proper Lorentz group. 120
2. The relation between the representations of the groups $\mathfrak{G}_+$ and $\mathfrak{U}$. 126
3. The spinor representations of the group $\mathfrak{U}$. 126
4. The infinitesimal operators of a spinor representation. 129
5. The irreducibility of a spinor representation. 132
6. The infinitesimal operators of a spinor representation with respect to a canonical basis. 133

10. PRINCIPAL SERIES OF REPRESENTATIONS OF THE GROUP $\mathfrak{A}$ 138
1. Some subgroups of the group $\mathfrak{A}$. 138
2. Canonical decomposition of the elements of the group $\mathfrak{A}$. 139
3. Residue classes with respect to K. 139
4. Parametrization of the space $\tilde{Z}$. 141
5. Invariant integral on the group Z. 142
6. The definition of the representations of the principal series. 144
7. Irreducibility of the representations of the principal series. 151

11. DESCRIPTION OF THE REPRESENTATIONS OF THE PRINCIPAL SERIES AND OF SPINOR REPRESENTATIONS BY MEANS OF THE UNITARY GROUP 154
1. A description of the space $\tilde{Z}$ in terms of the unitary subgroup. 154
2. The space $L_2^m(\mathfrak{U})$. 156
3. The realization of the representation of the principal series in the space $L_2^m(\mathfrak{U})$. 157
4. The representations S_k, contained in $\mathfrak{S}_{m,\rho}$. 159
5. Elementary spherical functions. 163
6. Infinitesimal operators of the representation $\mathfrak{S}_{m,\rho}$ in a canonical basis. 166
7. The case of spinor representations. 169

12. COMPLEMENTARY SERIES OF REPRESENTATIONS OF THE GROUP $\mathfrak{A}$ 170
1. Statement of the problem of complementary series. 170
2. The condition for positive definiteness. 174
3. The spaces $\mathfrak{H}_\sigma$ and H_σ. 179
4. A description of the representations of the complementary series in the space $\mathfrak{H}_\sigma$. 180
5. A description of the representations of the complementary series with the aid of the unitary subgroup. 182
6. The representations S_k, contained in $\mathfrak{D}_\sigma$. 185
7. The elementary spherical functions of the representations of the complementary series. 185
8. The infinitesimal operators of the representations $\mathfrak{D}_\sigma$ in a canonical basis. 186

13. THE TRACE OF A REPRESENTATION OF THE PRINCIPAL OR COMPLEMENTARY SERIES 188
1. An invariant integral on the group $\mathfrak{A}$. 188
2. Invariant integrals on the group K. 192
3. Some integral relations. 193
4. The group ring of the group $\mathfrak{A}$. 197
5. The relation between the representations of the group $\mathfrak{A}$ and its group ring. 199
6. The case of a unitary representation of the group $\mathfrak{A}$. 204

7. The trace of a representation of the principal series. 205
8. The trace of a representation of the complementary series. 208

14. An Analogue of Plancherel's Formula 210

1. Statement of the problem. 210
2. Some subgroups of the group K. 216
3. Canonical decomposition of the elements of the group K. 217
4. Some integral relations. 218
5. Some auxiliary functions and relations between them. 219
6. The derivation of an analogue of Plancherel's formula. 223
7. The inverse formulae. 227
8. The decomposition of the regular representation of the group $\mathfrak{U}$ into irreducible representations. 232

15. A Description of all the Completely Irreducible Representations of the Proper Lorentz Group 234

1. Conjugate representations. 235
2. The operators E_{jl}^{k}. 238
3. Equivalence of representations. 238
4. Completely irreducible representations. 242
5. The operators e_{jl}^{k}. 248
6. The ring X_j^k. 249
7. The relation between the representations of the rings X and X_j^k. 251
8. The commutativity of the rings X_j^k. 254
9. A criterion of equivalence. 258
10. The functional $\lambda(x)$ in the case of an irreducible representation of the principal series. 261
11. The functions $B_\nu(\varepsilon)$. 262
12. The ring $\tilde{\mathfrak{B}}_j^k$. 272
13. The general form of the functional $\lambda(b)$. 274
14. The general form of the linear multiplicative functional $\lambda(B)$ in the ring $\mathfrak{B}_j^k$. 279
15. The complete series of completely irreducible representations of the group $\mathfrak{A}$. 285
16. A fundamental theorem. 295

16. Description of all the Completely Irreducible Representations of the Complete Lorentz Group 297

1. Statement of the problem. 297
2. The fundamental properties of the operator S. 297
3. The group ring of the group $\mathfrak{G}_0$. 300
4. Induced representations. 301
5. Description of the completely irreducible representations of the ring $\mathfrak{C}_j^k$. 302

6. Realizations of the completely irreducible representations of the group $\mathfrak{G}_0$. 312
7. A fundamental theorem. 323

CHAPTER IV. INVARIANT EQUATIONS

17. Equations Invariant with Respect to Rotations of Three-dimensional Space 327
1. A general definition of quantities. 327
2. The concept of an equation invariant with respect to a representation of the group G_0. 328
3. Conditions of invariance. 330
4. Conditions of invariance in infinitesimal form. 330
5. General form of the operators L_1, L_2, L_3. 334

18. Equations Invariant with Respect to Proper Lorentz Transformations 347
1. General linear representations of the proper Lorentz group in infinitesimal form. 347
2. Some special cases of representations of the group $\mathfrak{G}_+$. 355
3. The concept of an equation invariant with respect to proper Lorentz transformations. 356
4. The general form of an equation invariant with respect to the transformations of the group $\mathfrak{G}_+$. 358

19. Equations Invariant with Respect to Transformations of the Complete Lorentz Group 373
1. General linear representations of the complete Lorentz group in infinitesimal form. 373
2. A description of the equations invariant with respect to the complete Lorentz group. 378

20. Equations Derived from an Invariant Lagrangian Function 382
1. Invariant bilinear forms. 382
2. Lagrangian functions. 396
3. The definition of rest mass and spin. 400
4. Conditions of definiteness of density of charge and energy. 403
5. The case of finite-dimensional equations. 411
6. Examples of invariant equations. 417

APPENDIX 423

REFERENCES 440

INDEX 445

Volumes Published in the Series in Pure and Applied Mathematics 449