

# Contents

PREFACE . . . . .	VII
OUTLINE OF THE BOOK . . . . .	XV
NOTATIONS . . . . .	XIX
CHAPTER 1	
LIE ALGEBRAS	
§ 1. Basic Concepts and General Properties . . . . .	1
§ 2. Solvable, Nilpotent, Semisimple and Simple Lie Algebras . . . . .	10
§ 3. The Structure of Lie Algebras . . . . .	17
§ 4. Classification of Simple, Complex Lie Algebras . . . . .	20
§ 5. Classification of Simple, Real Lie Algebras . . . . .	29
§ 6. The Gauss, Cartan and Iwasawa Decompositions . . . . .	37
§ 7. An Application. On Unification of the Poincaré Algebra and Internal Symmetry Algebra . . . . .	43
§ 8. Contraction of Lie Algebras . . . . .	44
§ 9. Comments and Supplements . . . . .	46
§ 10. Exercises . . . . .	48
CHAPTER 2	
TOPOLOGICAL GROUPS	
§ 1. Topological Spaces . . . . .	52
§ 2. Topological Groups . . . . .	61
§ 3. The Haar Measure . . . . .	67
§ 4. Comments and Supplements . . . . .	70
§ 5. Exercises . . . . .	71
CHAPTER 3	
LIE GROUPS	
§ 1. Differentiable Manifolds . . . . .	75
§ 2. Lie Groups . . . . .	81
§ 3. The Lie Algebra of a Lie Group . . . . .	85
§ 4. The Direct and Semidirect Products . . . . .	95
§ 5. Levi-Malcev Decomposition . . . . .	98
§ 6. Gauss, Cartan, Iwasawa and Bruhat Global Decompositions . . . . .	100
§ 7. Classification of Simple Lie Groups . . . . .	106
§ 8. Structure of Compact Lie Groups . . . . .	108

§ 9. Invariant Metric and Invariant Measure on Lie Groups . . . . .	109
§ 10. Comments and Supplements . . . . .	111
§ 11. Exercises . . . . .	114

## CHAPTER 4

## HOMOGENEOUS AND SYMMETRIC SPACES

§ 1. Homogeneous Spaces . . . . .	123
§ 2. Symmetric Spaces . . . . .	124
§ 3. Invariant and Quasi-Invariant Measures on Homogeneous Spaces	128
§ 4. Comments and Supplements . . . . .	132
§ 5. Exercises . . . . .	132

## CHAPTER 5

## GROUP REPRESENTATIONS

§ 1. Basic Concepts . . . . .	134
§ 2. Equivalence of Representations . . . . .	139
§ 3. Irreducibility and Reducibility . . . . .	141
§ 4. Cyclic Representations . . . . .	145
§ 5. Tensor Product of Representations . . . . .	147
§ 6. Direct Integral Decomposition of Unitary Representations . . . .	150
§ 7. Comments and Supplements . . . . .	156
§ 8. Exercises . . . . .	

## CHAPTER 6

## REPRESENTATIONS OF COMMUTATIVE GROUPS

§ 1. Irreducible Representations and Characters . . . . .	159
§ 2. Stone and SNAG Theorems . . . . .	160
§ 3. Comments and Supplements . . . . .	163
§ 4. Exercises . . . . .	164

## CHAPTER 7

## REPRESENTATIONS OF COMPACT GROUPS

§ 1. Basic Properties of Representations of Compact Groups . . . . .	166
§ 2. Peter-Weyl and Weyl Approximation Theorems . . . . .	172
§ 3. Projection Operators and Irreducible Representations . . . . .	177
§ 4. Applications . . . . .	179
§ 5. Representations of Finite Groups . . . . .	186
§ 6. Comments and Supplements . . . . .	195
§ 7. Exercises . . . . .	197

## CHAPTER 8

## FINITE-DIMENSIONAL REPRESENTATIONS OF LIE GROUPS

§ 1. General Properties of Representations of Solvable and Semisimple Lie Groups . . . . .	199
---	-----

§ 2. Induced Representations of Lie Groups . . . . .	205
§ 3. The Representations of $GL(n, C)$ , $GL(n, R)$ , $U(p, q)$ , $U(n)$ , $SL(n, C)$ , $SL(n, R)$ , $SU(p, q)$ , and $SU(n)$ . . . . .	213
§ 4. The Representations of the Symplectic Groups $Sp(n, C)$ , $Sp(n, R)$ and $Sp(n)$ . . . . .	217
§ 5. The Representations of Orthogonal Groups $SO(n, C)$ , $SO(p, q)$ , $SO^*(n)$ , and $SO(n)$ . . . . .	219
§ 6. The Fundamental Representations . . . . .	223
§ 7. Representations of Arbitrary Lie Groups . . . . .	225
§ 8. Further Results and Comments . . . . .	227
§ 9. Exercises . . . . .	238

## CHAPTER 9

## TENSOR OPERATORS, ENVELOPING ALGEBRAS AND ENVELOPING FIELDS

§ 1. The Tensor Operators . . . . .	242
§ 2. The Enveloping Algebra . . . . .	249
§ 3. The Invariant Operators . . . . .	251
§ 4. Casimir Operators for Classical Lie Group . . . . .	254
§ 5. The Enveloping Field . . . . .	266
§ 6. Further Results and Comments . . . . .	273
§ 7. Exercises . . . . .	275

## CHAPTER 10

THE EXPLICIT CONSTRUCTION OF FINITE-DIMENSIONAL IRREDUCIBLE  
REPRESENTATIONS

§ 1. The Gel'fand-Zetlin Method . . . . .	277
§ 2. The Tensor Method . . . . .	291
§ 3. The Method of Harmonic Functions . . . . .	302
§ 4. The Method of Creation and Annihilation Operators . . . . .	309
§ 5. Comments and Supplements . . . . .	312
§ 6. Exercises . . . . .	314

## CHAPTER 11

REPRESENTATION THEORY OF LIE AND ENVELOPING ALGEBRAS BY UNBOUNDED  
OPERATORS: ANALYTIC VECTORS AND INTEGRABILITY

§ 1. Representations of Lie Algebras by Unbounded Operators . . . . .	318
§ 2. Representations of Enveloping Algebras by Unbounded Operators . . . . .	323
§ 3. Analytic Vectors and Analytic Dominance . . . . .	331
§ 4. Analytic Vectors for Unitary Representations of Lie Groups . . . . .	344
§ 5. Integrability of Representations of Lie Algebras . . . . .	348
§ 6. $FS^3$ -Theory of Integrability of Lie Algebras Representations . . . . .	352
§ 7. The 'Heat Equation' on a Lie Group and Analytic Vectors . . . . .	358

§ 8. Algebraic Construction of Irreducible Representations . . . . .	365
§ 9. Comments and Supplements . . . . .	372
§ 10. Exercises . . . . .	373

## CHAPTER 12

## QUANTUM DYNAMICAL APPLICATIONS OF LIE ALGEBRA REPRESENTATIONS

§ 1. Symmetry Algebras in Hamiltonian Formulation . . . . .	378
§ 2. Dynamical Lie Algebras . . . . .	382
§ 3. Exercises . . . . .	386

## CHAPTER 13

## GROUP THEORY AND GROUP REPRESENTATIONS IN QUANTUM THEORY

§ 1. Group Representations in Physics . . . . .	392
§ 2. Kinematical Postulates of Quantum Theory . . . . .	394
§ 3. Symmetries of Physical Systems . . . . .	406
§ 4. Dynamical Symmetries of Relativistic and Non-Relativistic Systems	412
§ 5. Comments and Supplements . . . . .	417
§ 6. Exercises . . . . .	418

## CHAPTER 14

## HARMONIC ANALYSIS ON LIE GROUPS. SPECIAL FUNCTIONS AND GROUP REPRESENTATIONS

§ 1. Harmonic Analysis on Abelian and Compact Lie Groups . . . . .	421
§ 2. Harmonic Analysis on Unimodular Lie Groups . . . . .	423
§ 3. Harmonic Analysis on Semidirect Product of Groups . . . . .	431
§ 4. Comments and Supplements . . . . .	435
§ 5. Exercises . . . . .	

## CHAPTER 15

## HARMONIC ANALYSIS ON HOMOGENEOUS SPACES

§ 1. Invariant Operators on Homogeneous Spaces . . . . .	439
§ 2. Harmonic Analysis on Homogeneous Spaces . . . . .	441
§ 3. Harmonic Analysis on Symmetric Spaces Associated with Pseudo-Orthogonal Groups $SO(p, q)$ . . . . .	446
§ 4. Generalized Projection Operators . . . . .	459
§ 5. Comments and Supplements . . . . .	466
§ 6. Exercises . . . . .	470

## CHAPTER 16

## INDUCED REPRESENTATIONS

§ 1. The Concept of Induced Representations . . . . .	473
§ 2. Basic Properties of Induced Representation . . . . .	487

§ 3. Systems of Imprimitivity . . . . .	493
§ 4. Comments and Supplements . . . . .	501
§ 5. Exercises . . . . .	493

## CHAPTER 17

## INDUCED REPRESENTATIONS OF SEMIDIRECT PRODUCTS

§ 1. Representation Theory of Semidirect Products . . . . .	503
§ 2. Induced Unitary Representations of the Poincaré Group . . . . .	513
§ 3. Representation of the Extended Poincaré Group . . . . .	525
§ 4. Indecomposable Representations of Poincaré Group . . . . .	527
§ 5. Comments and Supplements . . . . .	536
§ 6. Exercises . . . . .	537

## CHAPTER 18

## FUNDAMENTAL THEOREMS OF INDUCED REPRESENTATIONS

§ 1. The Induction-Reduction Theorem . . . . .	540
§ 2. Tensor-Product Theorem . . . . .	546
§ 3. The Frobenius Reciprocity Theorem . . . . .	549
§ 4. Comments and Supplements . . . . .	553
§ 5. Exercises . . . . .	553

CHAPTER 19<sup>1</sup>

## INDUCED REPRESENTATIONS OF SEMISIMPLE LIE GROUPS

§ 1. Induced Representations of Semisimple Lie Groups . . . . .	555
§ 2. Properties of the Group $SL(n, C)$ and Its Subgroups . . . . .	559
§ 3. The Principal Nondegenerate Series of Unitary Representations of $SL(n, C)$ . . . . .	560
§ 4. Principal Degenerate Series of $SL(n, C)$ . . . . .	567
§ 5. Supplementary Nondegenerate and Degenerate Series . . . . .	570
§ 6. Comments and Supplements . . . . .	577
§ 7. Exercises . . . . .	578

## CHAPTER 20

## APPLICATIONS OF INDUCED REPRESENTATIONS

§ 1. The Relativistic Position Operator . . . . .	581
§ 2. The Representations of the Heisenberg Commutation Relations . . . . .	588
§ 3. Comments and Supplements . . . . .	591
§ 4. Exercises . . . . .	593

## CHAPTER 21

## GROUP REPRESENTATIONS IN RELATIVISTIC QUANTUM THEORY

§ 1. Relativistic Wave Equations and Induced Representations . . . . .	596
§ 2. Finite Component Relativistic Wave Equations . . . . .	601

§ 3. Infinite Component Wave Equations . . . . .	609
§ 4. Group Extensions and Applications . . . . .	619
§ 5. Space-Time and Internal Symmetries . . . . .	626
§ 6. Comments and Supplements . . . . .	630
§ 7. Exercises . . . . .	636
APPENDIX A	
ALGEBRA, TOPOLOGY, MEASURE AND INTEGRATION THEORY . . . . .	637
APPENDIX B	
FUNCTIONAL ANALYSIS	
§ 1. Closed, Symmetric and Self-Adjoint Operators in Hilbert Space . . . . .	641
§ 2. Integration of Vector and Operator Functions . . . . .	645
§ 3. Spectral Theory of Operators . . . . .	649
§ 4. Functions of Self-Adjoint Operators . . . . .	662
§ 5. Essentially Self-Adjoint Operators . . . . .	663
BIBLIOGRAPHY . . . . .	667
LIST OF IMPORTANT SYMBOLS . . . . .	703
AUTHOR INDEX . . . . .	706
SUBJECT INDEX . . . . .	710