

# Contents

Introduction	1
<b>CHAPTER I</b>	
<b>Some Homological Algebra</b>	<b>4</b>
0. Review of Chain Complexes	4
1. Free Resolutions	10
2. Group Rings	12
3. $G$ -Modules	13
4. Resolutions of $\mathbb{Z}$ Over $\mathbb{Z}G$ via Topology	14
5. The Standard Resolution	18
6. Periodic Resolutions via Free Actions on Spheres	20
7. Uniqueness of Resolutions	21
8. Projective Modules	26
Appendix. Review of Regular Coverings	31
<b>CHAPTER II</b>	
<b>The Homology of a Group</b>	<b>33</b>
1. Generalities	33
2. Co-invariants	34
3. The Definition of $H_*G$	35
4. Topological Interpretation	36
5. Hopf's Theorems	41
6. Functoriality	48
7. The Homology of Amalgamated Free Products	49
Appendix. Trees and Amalgamations	52
<b>CHAPTER III</b>	
<b>Homology and Cohomology with Coefficients</b>	<b>55</b>
0. Preliminaries on $\otimes_G$ and $\text{Hom}_G$	55
1. Definition of $H_*(G, M)$ and $H^*(G, M)$	56

2. Tor and Ext	60
3. Extension and Co-extension of Scalars	62
4. Injective Modules	65
5. Induced and Co-induced Modules	67
6. $H_*$ and $H^*$ as Functors of the Coefficient Module	71
7. Dimension Shifting	74
8. $H_*$ and $H^*$ as Functors of Two Variables	78
9. The Transfer Map	80
10. Applications of the Transfer	83

## CHAPTER IV

Low Dimensional Cohomology and Group Extensions	86
1. Introduction	86
2. Split Extensions	87
3. The Classification of Extensions with Abelian Kernel	91
4. Application: $p$ -Groups with a Cyclic Subgroup of Index $p$	97
5. Crossed Modules and $H^3$ (Sketch)	102
6. Extensions With Non-Abelian Kernel (Sketch)	104

## CHAPTER V

Products	107
1. The Tensor Product of Resolutions	107
2. Cross-products	108
3. Cup and Cap Products	109
4. Composition Products	114
5. The Pontryagin Product	117
6. Application: Calculation of the Homology of an Abelian Group	121

## CHAPTER VI

Cohomology Theory of Finite Groups	128
1. Introduction	128
2. Relative Homological Algebra	129
3. Complete Resolutions	131
4. Definition of $\hat{H}^*$	134
5. Properties of $\hat{H}^*$	136
6. Composition Products	142
7. A Duality Theorem	144
8. Cohomologically Trivial Modules	148
9. Groups with Periodic Cohomology	153

## CHAPTER VII

Equivariant Homology and Spectral Sequences	161
1. Introduction	161
2. The Spectral Sequence of a Filtered Complex	161

3. Double Complexes	164
4. Example: The Homology of a Union	166
5. Homology of a Group with Coefficients in a Chain Complex	168
6. Example: The Hochschild–Serre Spectral Sequence	171
7. Equivariant Homology	172
8. Computation of $d^1$	175
9. Example: Amalgamations	178
10. Equivariant Tate Cohomology	180

## CHAPTER VIII

## Finiteness Conditions

183

1. Introduction	183
2. Cohomological Dimension	184
3. Serre's Theorem	190
4. Resolutions of Finite Type	191
5. Groups of Type $FP_n$	197
6. Groups of Type $FP$ and $FL$	199
7. Topological Interpretation	205
8. Further Topological Results	210
9. Further Examples	213
10. Duality Groups	219
11. Virtual Notions	225

## CHAPTER IX

## Euler Characteristics

230

1. Ranks of Projective Modules: Introduction	230
2. The Hattori–Stallings Rank	231
3. Ranks Over Commutative Rings	235
4. Ranks Over Group Rings; Swan's Theorem	239
5. Consequences of Swan's Theorem	242
6. Euler Characteristics of Groups: The Torsion-Free Case	246
7. Extension to Groups with Torsion	249
8. Euler Characteristics and Number Theory	253
9. Integrality Properties of $\chi(\Gamma)$	257
10. Proof of Theorem 9.3; Finite Group Actions	258
11. The Fractional Part of $\chi(\Gamma)$	261
12. Acyclic Covers; Proof of Lemma 11.2	265
13. The $p$ -Fractional Part of $\chi(\Gamma)$	266
14. A Formula for $\chi_r(\mathcal{A})$	270

## CHAPTER X

## Farrell Cohomology Theory

273

1. Introduction	273
2. Complete Resolutions	273
3. Definition and Properties of $\hat{H}^*(\Gamma)$	277

4. Equivariant Farrell Cohomology	281
5. Cohomologically Trivial Modules	287
6. Groups with Periodic Cohomology	288
7. $\hat{H}^*(\Gamma)$ and the Ordered Set of Finite Subgroups of $\Gamma$	291
References	295
Notation Index	301
Index	303