Contents

Acknowledgments vii

Introduction xi

About This Book xv

1 Cardinal and Ordinal Numbers 1
1.1 Countable Sets 1
1.2 Order of Infinity 4
1.3 The Axiom of Choice 7
1.4 More on Equinumerosity 11
1.5 Arithmetic of Cardinal Numbers 13
1.6 Well-Ordered Sets 15
1.7 Transfinite Induction 18
1.8 Ordinal Numbers 21
1.9 Alephs 24
1.10 Trees 26
1.11 Induction on Trees 29
1.12 The Souslin Operation 31
1.13 Idempotence of the Souslin Operation 34

2 Topological Preliminaries 39
2.1 Metric Spaces 39
2.2 Polish Spaces 52
2.3 Compact Metric Spaces 57
2.4 More Examples 63

2.5 The Baire Category Theorem 69
2.6 Transfer Theorems 74

3 Standard Borel Spaces **81**
3.1 Measurable Sets and Functions 81
3.2 Borel-Generated Topologies 91
3.3 The Borel Isomorphism Theorem 94
3.4 Measures 100
3.5 Category 107
3.6 Borel Pointclasses 115

4 Analytic and Coanalytic Sets **127**
4.1 Projective Sets 127
4.2 $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ Complete Sets 135
4.3 Regularity Properties 141
4.4 The First Separation Theorem 147
4.5 One-to-One Borel Functions 150
4.6 The Generalized First Separation Theorem 155
4.7 Borel Sets with Compact Sections 157
4.8 Polish Groups 160
4.9 Reduction Theorems 164
4.10 Choquet Capacitability Theorem 172
4.11 The Second Separation Theorem 175
4.12 Countable-to-One Borel Functions 178

5 Selection and Uniformization Theorems **183**
5.1 Preliminaries 184
5.2 Kuratowski and Ryll-Nardzewski's Theorem 189
5.3 Dubins – Savage Selection Theorems 194
5.4 Partitions into Closed Sets 195
5.5 Von Neumann's Theorem 198
5.6 A Selection Theorem for Group Actions 200
5.7 Borel Sets with Small Sections 204
5.8 Borel Sets with Large Sections 206
5.9 Partitions into G_δ Sets 212
5.10 Reflection Phenomenon 216
5.11 Complementation in Borel Structures 218
5.12 Borel Sets with σ-Compact Sections 219
5.13 Topological Vaught Conjecture 227
5.14 Uniformizing Coanalytic Sets 236

References **241**

Glossary **251**

Index **253**