# Contents

_ /			٠
Preface >	(I	ı	۱

#### Notation xviii

- 1 What is Combinatorics? 1
  - 1.1 The Three Problems of Combinatorics 1
  - 1.2 The History and Applications of Combinatorics 8

References for Chapter 1 11

## PART I The Basic Tools of Combinatorics 13

- 2 Basic Counting Rules (co-authored by Helen Marcus-Roberts) 13
  - 2.1 The Product Rule 13
  - 2.2 The Sum Rule 19
  - 2.3 Permutations 20
  - 2.4 Complexity of Computation 22
  - 2.5 r-Permutations 26
  - 2.6 Subsets 27
  - 2.7 Combinations 28
  - 2.8 Pascal's Triangle 32
  - 2.9 Probability 33
  - 2.10 Sampling with Replacement 38

#### 2.11 Occupancy Problems 40

2.11.1 The Types of Occupancy Problems, 40 2.11.2 Case 2: Indistinguishable Balls and Distinguishable Cells, 42 2.11.3 Case 3: Distinguishable Balls and Indistinguishable Cells, 43 2.11.4 Case 4: Indistinguishable Balls and Indistinguishable Cells, 43 2.11.5 Examples, 44

#### 2.12 Multinomial Coefficients 47

- 2.12.1 Occupancy Problems with a Specified Distribution, 47 2.12.2 Permutations with Classes of Indistinguishable Objects, 49
- 2.13 Complete Digest by Enzymes 51
- 2.14 Permutations with Classes of Indistinguishable Objects Revisited 54
- 2.15 The Binomial Expansion 56
- 2.16 Power in Simple Games 58
  - 2.16.1 Examples of Simple Games, 58 2.16.2 The Shapley-Shubik Power Index, 60 2.16.3 The U.N. Security Council, 62 2.16.4 Bicameral Legislatures, 63 2.16.5 Characteristic Functions, 64
- 2.17 An Algorithm for Generating Permutations 65
- 2.18 Good Algorithms 67
  - 2.18.1 Asymptotic Analysis, 67 2.18.2 NP-Complete Problems, 70

Additional Exercises for Chapter 2 71 References for Chapter 2 73

## 3 Introduction to Graph Theory 76

#### 3.1 Fundamental Concepts 76

3.1.1 Some Examples, 76 3.1.2 Definition of Digraph and Graph, 80 3.1.3 Labeled Digraphs and the Isomorphism Problem, 83

#### 3.2 Connectedness 88

3.2.1 Reaching in Digraphs, 88 3.2.2 Joining in Graphs, 89 3.2.3 Strongly Connected Digraphs and Connected Graphs, 90 3.2.4 Subgraphs, 91 3.2.5 Connected Components, 92

## 3.3 Graph Coloring and Its Applications 97

3.3.1 Some Applications, 97 3.3.2 Kuratowski's Theorem, 102 3.3.3 Calculating the Chromatic Number, 103 3.3.4 2-Colorable Graphs, 104

## 3.4 Chromatic Polynomials 111

3.4.1 Definitions and Examples, 111 3.4.2 Reduction Theorems, 114 3.4.3 Properties of Chromatic Polynomials, 118

#### 3.5 Trees 122

3.5.1 Definition of a Tree and Examples, 122 3.5.2 Properties of Trees, 122 3.5.3 Proof of Theorem 3.10, 124 3.5.4 Spanning Trees, 125 3.5.5 Proof of Theorem 3.11 and a Related Result, 127 3.5.6 Chemical Bonds and the Number of Trees, 128

vi

- 3.6 Applications of Trees to Searching and Sorting Problems 133
  - 3.6.1 Search Trees, 133 3.6.2 Proof of Theorem 3.19, 134 3.6.3 Sorting, 135
- 3.7 Representing a Graph in the Computer 141 References for Chapter 3 145

## PART II The Counting Problem 149

- 4 Generating Functions and Their Applications 149
  - 4.1 Definition 149
    - 4.1.1 Power Series, 150 4.1.2 Generating Functions, 152
  - 4.2 Operating on Generating Functions 159
  - 4.3 Applications to Counting 165
    - 4.3.1 Sampling Problems, 165 4.3.2 A Comment on Occupancy Problems, 169
  - 4.4 The Binomial Theorem 172
  - 4.5 Exponential Generating Functions and Generating Functions for Permutations 177
    - 4.5.1 Definition of Exponential Generating Function, 177 4.5.2 Applications to Counting Permutations, 179 4.5.3 Distributions of Distinguishable Balls into Indistinguishable Cells, 182
  - 4.6 Probability Generating Functions 185
  - 4.7 The Coleman and Banzhaf Power Indices 188

References for Chapter 4 192

- 5 Recurrence Relations 194
  - 5.1 Some Examples 194
    - 5.1.1 Some Simple Recurrences, 194 5.1.2 Fibonacci Numbers and Their Applications, 200
    - 5.1.3 Derangements, 203 5.1.4 Recurrences Involving More Than One Sequence, 206
  - 5.2 The Method of Characteristic Roots 210
    - 5.2.1 The Case of Distinct Roots, 210 5.2.2 Computation of the kth Fibonacci Number, 213
    - 5.2.3 The Case of Multiple Roots, 214
  - 5.3 Solving Recurrences Using Generating Functions 218
    - 5.3.1 The Method, 218 5.3.2 Derangements, 223 5.3.3 Simultaneous Equations for Generating Functions, 226
  - 5.4 Some Recurrences Involving Convolutions 230
    - 5.4.1 The Number of Simple, Ordered, Rooted Trees, 230 5.4.2 The Ways to Multiply a Sequence of Numbers in a Computer, 234 5.4.3 Secondary Structure in RNA, 236 5.4.4 Organic Compounds Built Up from Benzene Rings, 239
  - 5.5 Divide-and-Conquer Algorithms 245 References for Chapter 5 250

## 6 The Principle of Inclusion and Exclusion 252

#### 6.1 The Principle and Some of Its Applications 252

6.1.1 Some Simple Examples, 252 6.1.2 Proof of Theorem 6.1, 255 6.1.3 The Sieve of Erastothenes, 256 6.1.4 The Probabilistic Case, 258 6.1.5 The Occupancy Problem with Distinguishable Balls and Cells, 258 6.1.6 Chromatic Polynomials, 259 6.1.7 Derangements, 262 6.1.8 Counting Combinations, 263 6.1.9 Rook Polynomials, 265

#### 6.2 The Number of Objects Having Exactly m Properties 270

6.2.1 The Main Result and Its Applications, 270 6.2.2 Proofs of Theorems 6.3 and 6.4, 276

References for Chapter 6 280

## 7 The Polya Theory of Counting 281

#### 7.1 Equivalence Relations 281

7.1.1 Definition and Examples, 281 7.1.2 Equivalence Classes, 285

#### 7.2 Permutation Groups 287

7.2.1 Definition of a Permutation Group, 287 7.2.2 The Equivalence Relation Induced by a Permutation Group, 290

- 7.3 Burnside's Lemma 293
- 7.4 Distinct Colorings 297

7.4.1 Definition of a Coloring, 297 7.4.2 Equivalent Colorings, 299 7.4.3 The Case of Switching Functions, 300

#### 7.5 The Cycle Index 304

7.5.1 Permutations as Products of Cycles, 304 7.5.2 A Special Case of Polya's Theorem, 306 7.5.3 The Case of Switching Functions, 307 7.5.4 The Cycle Index of a Permutation Group, 307 7.5.5 Proof of Theorem 7.5, 309

#### 7.6 Polya's Theorem 311

7.6.1 The Inventory of Colorings, 311 7.6.2 Computing the Pattern Inventory, 313 7.6.3 The Case of Switching Functions, 314 7.6.4 Proof of Polya's Theorem, 315

References for Chapter 7 317

## PART III The Existence Problem 319

## 8 The Pigeonhole Principle and its Generalizations 319

## 8.1 Pigeons in Holes 319

8.1.1 The Simplest Version of the Pigeonhole Principle, 319 8.1.2 Some Generalizations and Applications of the Pigeonhole Principle, 321

#### 8.2 Ramsey Theory 325

8.2.1 Ramsey's Theorem, 325 8.2.2 The Ramsey Numbers R(p,q), 327 8.2.3 Bounds on Ramsey Numbers, 329 8.2.4 Generalizations of Ramsey's Theorem, 331 8.2.5 Graph Ramsey Numbers, 333

#### 8.3 Applications of Ramsey Theory 336

8.3.1 Convex m-gons, 336 8.3.2 Confusion Graphs for Noisy Channels, 339 8.3.3 Design of Packet-Switched Networks, 341 8.3.4 Information Retrieval, 343 8.3.5 The Dimension of Partial Orders: A Decisionmaking Application, 346

References for Chapter 8 353

## 9 Experimental Design 356

- 9.1 Block Designs 356
- 9.2 Latin Squares 360
  - 9.2.1 Some Examples, 360 9.2.2 Orthogonal Latin Squares, 362 9.2.3 Existence Results for Orthogonal Families, 365
- 9.3 Finite Fields and Complete Orthogonal Families of Latin Squares 373
  - 9.3.1 Modular Arithmetic, 373 9.3.2 The Finite Fields  $GF(p^k)$ , 374 9.3.3 Construction of a Complete Orthogonal Family of  $n \times n$  Latin Squares if n is a Power of a Prime, 377 9.3.4 Justification of the Construction of a Complete Orthogonal Family if  $n = p^k$ , 378
- 9.4 Balanced Incomplete Block Designs, 381

9.4.1  $(b, v, r, k, \lambda)$ -designs, 381 9.4.2 Necessary Conditions for the Existence of  $(b, v, r, k, \lambda)$ -designs, 385 9.4.3 Proof of Fisher's Inequality, 387 9.4.4 Steiner Triple Systems, 388 9.4.5 Symmetric Balanced Incomplete Block Designs, 390 9.4.6 Building New  $(b, v, r, k, \lambda)$ -designs from Existing Ones, 392

## 9.5 Finite Projective Planes 396

9.5.1 Basic Properties, 396 9.5.2 Projective Planes, Latin Squares, and  $(v, k, \lambda)$ -designs, 400

# References for Chapter 9 404

Coding Theory

10

- 10.1 Information Transmission 406
- 10.2 Encoding and Decoding 407
- 10.3 Error-Correcting Codes 410

10.3.1 Error Correction and Hamming Distance, 410 10.3.2 The Hamming Bound, 413 10.3.3 The Probability of Error, 414

#### 10.4 Linear Codes 416

10.4.1 Generator Matrices, 416 10.4.2 Error Correction Using Linear Codes, 419 10.4.3 Hamming Codes, 423

10.5 The Use of Block Designs to Find Error-Correcting Codes 427

10.5.1 Hadamard Codes, 427 10.5.2 Constructing Hadamard Designs, 428 10.5.3 The Richest (n, d)-Codes, 433

References for Chapter 10 440

## 11 Existence Problems in Graph Theory 442

11.1 Depth-First Search: A Test for Connectedness 443

11.1.1 Depth-First Search, 443 11.1.2 The Computational Complexity of Depth-First Search, 445 11.1.3 A Formal Statement of the Algorithm, 445

11.2 The One-way Street Problem 448

11.2.1 Robbins' Theorem, 448 11.2.2 A Depth-First Search Algorithm, 452 11.2.3 Efficient One-way Street Assignments, 453

11.3 Eulerian Chains and Paths 457

11.3.1 The Königsberg Bridge Problem, 457 11.3.2 An Algorithm for Finding an Eulerian Closed Chain, 458 11.3.3 Further Results about Eulerian Chains and Paths, 460

11.4 Applications of Eulerian Chains and Paths 465

11.4.1 The "Chinese Postman" Problem, 465 11.4.2 Computer Graph Plotting, 467 11.4.3 Street Sweeping, 467 11.4.4 RNA Chains and Complete Digests, 469 11.4.5 A Coding Application, 470 11.4.6 De Bruijn Sequences and Telecommunications, 473

11.5 Hamiltonian Chains and Paths 478

11.5.1 Definitions, 478 11.5.2 Sufficient Conditions for the Existence of a Hamiltonian Circuit in a Graph, 480 11.5.3 Sufficient Conditions for the Existence of a Hamiltonian Cycle in a Digraph, 483

11.6 Applications of Hamiltonian Chains and Paths 486

11.6.1 Tournaments, 486 11.6.2 Topological Sorting, 489 11.6.3 Scheduling Problems in Operations Research, 490 11.6.4 Facilities Design, 490

References for Chapter 11 494

# PART IV Combinatorial Optimization 497

## 12 Matching and Covering 497

12.1 Some Matching Problems 497

12.2 Some Existence Results: Bipartite Matching and Systems of Distinct Representatives 501

12.2.1 Bipartite Matching, 501 12.2.2 Systems of Distinct Representatives, 504

12.3 The Existence of Perfect Matchings for Arbitrary Graphs 509

12.4 Maximum Matchings and Minimum Coverings 511

12.4.1 Vertex Coverings, 511 12.4.2 Edge Coverings, 513

- 12.5 Finding a Maximum Matching 515
  - 12.5.1 M-augmenting Chains, 515 12.5.2 Proof of Theorem 12.7, 516 12.5.3 An Algorithm for Finding a Maximum Matching, 518
- 12.6 Matching as Many Elements of X as Possible 522 References for Chapter 12 524
- 13 Optimization Problems for Graphs and Networks 526
  - 13.1 Minimum Spanning Trees 526
    - 13.1.1 Kruskal's Algorithm, 526 13.1.2 Proof of Theorem 13.1, 528 13.1.3 Prim's Algorithm, 529
  - 13.2 The Shortest Route Problem 532
    - 13.2.1 The Problem, 532 13.2.2 Dijkstra's Algorithm, 533 13.2.3 Applications to Scheduling Problems, 536 13.2.4 The "Chinese Postman" Problem Revisited, 537
  - 13.3 Network Flows 541
    - 13.3.1 The Maximum Flow Problem, 541 13.3.2 Cuts, 543 13.3.3 A Faulty Max Flow Algorithm, 544 13.3.4 Augmenting Chains, 545 13.3.5 The Max Flow Algorithm, 549 13.3.6 A Labeling Procedure for Finding Augmenting Chains, 550 13.3.7 Complexity of the Max Flow Algorithm, 553 13.3.8 Matchina Revisited, 554
  - 13.4 Minimum Cost Flow Problems 561
  - 13.4.1 Some Examples, 561 13.4.2 An Algorithm for the Optimal Assignment Problem, 565
    References for Chapter 13 570

Answers to Selected Exercises 572

Author Index 583

Subject Index 589