

CONTENTS

1. Elements of category theory	1
Categories	1
Some basic category-theoretic notions	5
Limits and colimits	11
Existence of limits and colimits	16
Functors	18
Natural transformations and functor categories	21
Completeness of functor categories	26
Equivalence of categories	28
Adjunctions	30
Units and counits of adjunctions	32
Freedom and cofreedom	35
Uniqueness of adjoints	38
Preservation of limits and colimits	39
Cartesian closed categories	42
Reflective subcategories	45
Galois connections	46
2. Introducing toposes	49
Subobjects and subobject classifiers	49
Power objects: the concept of topos	58
\mathbf{Set}^c as a topos	60
Geometric morphisms	65
3. Local set theories	68
Local languages and local set theories	68
Logic in a local set theory	73
Set theory in a local language	83
The category of sets determined by a local set theory	85
Interpreting a local language in a topos: the soundness theorem	91
The completeness theorem	103
The equivalence theorem	105
Translations and logical functors	113
Adjoining indeterminates	115
Introduction of function values	117
4. Fundamental properties of toposes	120
Some fundamental properties of toposes	120
The structure of Ω and $\mathbf{Sub}(A)$ in a topos	127
Slicing a topos	131
Coproducts in a topos	136

Syntactic properties of local set theories versus essentially categorical properties of toposes	141
Full theories	150
Beth–Kripke–Joyal semantics	155
5. From logic to sheaves	162
Truth sets, modalities, and universal closure operations	162
Sheaves	171
The sheafification functor	180
Modalized toposes	189
Modal operators and sheaves in $\mathbf{Set}^{\text{cop}}$	194
Sheaves over locales and topological spaces	199
6. Locale-valued sets	202
Locale-valued sets	202
The topos of sheaves on a topological space	209
Decidable, subconstant, and fuzzy sets	210
Boolean extensions as toposes	215
7. Natural numbers and real numbers	220
Natural numbers in local set theories	220
Real numbers in local set theories	226
The free topos	233
8. Epilogue: the wider significance of topos theory	235
From set theory to topos theory	235
Some analogies with the theory of relativity	239
The negation of constancy	242
Appendix: Geometric theories and classifying toposes	246
Historical and bibliographical notes	255
References	258
Index of symbols	261
Index of terms	264