CONTENTS

Preface I

Chapter 1: *Combinatorial and algebraic foundations* 1

1.1 Matroids 1
1.2 Regular matroids 8
1.3 Examples for regular and non-regular matroids 16
1.4 Ordered semigroups $(H,*,\leq)$ 22
1.5 Examples of ordered semigroups 27

Chapter 2: *Maximal M-flows* 33

2.1 The Max M-Flow - Min Cocircuit Theorem and the Augmenting Circuit Algorithm 33
2.2 Decomposition of M-flows 41
2.3 Incremental matroids 45
2.4 Differences of M-flows 51

Chapter 3: *M-flows with lower capacities* 55

3.1 Determining maximal and minimal admissible M-flows 56
3.2 Admissibility Theorem and an algorithm for determining admissible M-flows 61
3.3 Decomposition of M-flows with lower capacity functions 65

Chapter 4: *Minimal cost M-flows* 69

4.1 Negative Circuit Algorithm 70
4.2 Shortest Augmenting Circuit Algorithm 75
4.3 Determining negative and shortest augmenting circuits 77
4.4 Compatibility Matroid Algorithm 82
4.5 Transformation Method 85

Chapter 5: *First application of the theory of M-flows: Algebraic flows in graphs* 94

5.1 Foundations 94

5.2 Maximal algebraic flows in graphs 97
5.2.1 Algebraic Max-Flow Min-Cut Theorem 99
5.2.2 Decomposition of algebraic flows 102
5.2.3 Incremental graphs and layered graph algorithms 102
5.2.4 Difference of flows in graphs 105

5.3 Flows in graphs with lower capacity function 106
5.3.1 Maximal and minimal admissible flows 107
5.3.2 Admissibility Theorem 110
5.3.3 Decomposition of flows in graphs 113

5.4 Minimal cost flows in graphs 115
5.4.1 Negative Circuit Algorithm 115
5.4.2 Shortest Augmenting Path Algorithm 117
5.4.3 Determining negative and shortest augmenting circuits by means of node functions 118
5.4.4 Compatibility graphs 121
5.4.5 Transformation Method 125

Chapter 6: *Second application of the theory of M-flows: Lexicographical M-flow problems* 128

6.1 The problem and its algebraic solution 128
6.2 Reduction to real-valued M-flow problems 131
6.3 Combinatorial comparison of the solution methods 138
6.4 Applications: Optimal cuts and shortest paths 140

Appendix 1: *Graph-theoretical definitions and denotations* 145

Appendix 2: *List of symbols* 147

References 151

Subject Index 163