## **Contents**

## PART I FOUNDATIONS

4.9 Miscellaneous Exercises

1.	Sta	tements and proofs			
	1.1	Some mathematical statements		3	
	1.2	How to do mathematics		4	
	1.3	Compound statements		6	
	1.4	Existential statements		7	
	1.5	Universal statements		8	
		Proof techniques		10	
	1.7	Miscellaneous Exercises		11	
2.	Set	notation			
	2.1	Sets of objects and numbers		12	
	2.2	Subsets		13	
	2.3	Union and intersection		14	
	2.4	Miscellaneous Exercises		15	
3.	The	logical framework			
	3.1	The basic logical operations: not, or, and		17	
	3.2	Logical equivalence		18	
	3.3	if - then		19	
	3.4	The converse statement		20	
	3.5	The contrapositive statement		21	
	3.6	Universal and existential quantifiers		22	
	3.7	Miscellaneous Exercises		24	
4.	Natural numbers				
	4.1	The 'laws of algebra'		25	
	4.2	Putting the natural numbers in order		26	
	4.3	The principle of induction		28	
	4.4	Summation formulae		29	
	4.5	Recursive definitions		31	
	4.6	Other forms of the principle of induction		32	
	4.7	Greatest and least members		34	
	4.8	How a conjecture becomes a theorem		35	

38

## x Contents

5.	. Functions			
	5.1		39	
	5.2		40	
	5.3	Composition of functions	42	
		Bijections and inverse functions	43	
		Miscellaneous Exercises	45	
6.	Hov	v to count		
	6.1	Counting as a bijection	46	
		The size of a set	47	
	6.3	A counting problem	49	
	6.4	Some applications of the pigeonhole principle	50	
	6.5	Infinite sets	51	
	6.6	Strange properties of infinite sets	53	
	6.7	Miscellaneous Exercises	55	
7.	Inte	gers		
		Negative numbers	56	
		Equivalence relations	56	
		Classification	58	
		Construction of the integers	60	
		Properties of the integers	61	
		Bounded subsets of $\mathbb{Z}$	63	
	7.7	Miscellaneous Exercises	64	
0	D: .:	Cathelith and decides a manual and		
8.		sibility and prime numbers	(5	
		Divisibility	65	
		Quotient and remainder	65	
		Representation of integers	66	
		The greatest common divisor Prime numbers	67 70	
			70 71	
	8.7	Existence and uniqueness of prime factorization Miscellaneous Exercises	73	
	0.7	Miscondificate Exercises	73	
9.	Fra	ctions and real numbers		
	9.1	Construction and properties of rational numbers	75	
	9.2	Density of fractions	76	
	9.3	Decimal representations of fractions	78	
	9.4	Real numbers	80	
	9.5	Approximations for real numbers	82	
	9.6	The greatest lower bound property	84	
	9.7	The real numbers are more plentiful than the rationals	85	
	9.8	Miscellaneous Exercises	87	

Contents xi	C	ont	en	ts	хi
-------------	---	-----	----	----	----

## PART II TECHNIQUES

	ciples of counting	
	The addition principle	91
	Counting sets of pairs	92
	Euler's function	95
	Functions, words, and selections	97
	Injections as ordered selections without repetition	99
	Permutations	100
10.7	Miscellaneous Exercises	103
11. Sub	sets and designs	
	Binomial numbers	105
	Unordered selections with repetition	108
11.3	The binomial theorem	110
11.4	The sieve principle	112
11.5	Some arithmetical applications	114
11.6	Designs	118
	t-designs	122
11.8	Miscellaneous Exercises	124
12. Par	tition, classification, and distribution	
12.1	Partitions of a set	126
	Classification and equivalence relations	128
12.3	Distributions and the multinomial numbers	129
12.4	Partitions of a positive integer	132
12.5	Classification of permutations	134
	Even and odd permutations	136
12.7	Miscellaneous Exercises	140
13. Mo	dular arithmetic	
13.1	Congruences	142
13.2	$\mathbb{Z}_m$ and its arithmetic	144
13.3	Invertible elements of $\mathbb{Z}_m$	147
13.4	Cyclic constructions for designs	149
	Latin squares	153
13.6	Miscellaneous Exercises	156
PAF	RT III ALGORITHMS AND GRAPHS	
1 <i>1</i> Δlα	orithms and their efficiency	
14.1		159
14.1		161
14.2	The state of the s	164
14.3		165
17.7	* * * * * * * * * * * * * * * * * * *	

xii	Con	tents	
	14.5	Efficiency of algorithms	168
		Growth rates: the <i>O</i> notation	170
		Comparison of algorithms	171
		Introduction to sorting algorithms	173
		Miscellaneous Exercises	176
15.	Grap	ohs	
	15.1	Graphs and their representation	178
	15.2	Isomorphism of graphs	179
	15.3	Degree	181
	15.4	Paths and cycles	183
	15.5	Trees	185
	15.6	Colouring the vertices of a graph	187
	15.7	The greedy algorithm for vertex-colouring	188
	15.8	Miscellaneous Exercises	191
16.	Tree	es, sorting, and searching	
		Counting the leaves on a rooted tree	193
	16.2	Trees and sorting algorithms	196
	16.3	• •	199
		Depth-first search	202
		Breadth-first search	205
		The shortest path problem	206
	16.7		208
17.	Bipa	artite graphs and matching problems	
	-	Relations and bipartite graphs	210
		Edge colourings of graphs	212
	17.3	Application of edge colouring to latin squares	213
		Matchings	216
		Maximum matchings	219
		<u> </u>	221
	17.7		223
18.	Diar	raphs, networks, and flows	
	18.1	Digraphs	225
	18.2	Networks and critical paths	227
	18.3	Flows and cuts	229
	18.4	The max-flow min-cut theorem	232
	18.5	The labelling algorithm for network flows	235
	18.6	Miscellaneous Exercises	239
19.	Rec	ursive techniques	
	19.1	Generalities about recursion	240
		Linear recursion	241

		Contents	xiii
	19.3 Recursive bisection		243
	19.4 Recursive optimization		245
	19.5 The framework of dynamic programming		248
	19.6 Examples of the dynamic programming method		250
	19.7 Miscellaneous Exercises		253
	PART IV ALGEBRAIC METHODS		
20.	Groups		
	20.1 The axioms for a group		259
	20.2 Examples of groups		260
	20.3 Basic algebra in groups		263
	20.4 The order of a group element		265
	20.5 Isomorphism of groups		266
	20.6 Cyclic groups		268
	20.7 Subgroups		270
	20.8 Cosets and Lagrange's theorem		273
	20.9 Characterization of cyclic groups		277
	20.10 Miscellaneous Exercises		279
21.	Groups of permutations		
	21.1 Definitions and examples		281
	21.2 Orbits and stabilizers		283
	21.3 The size of an orbit		285
	21.4 The number of orbits		288
	21.5 Representation of groups by permutations		290
	21.6 Applications to group theory		292
	21.7 Miscellaneous Exercises		295
22.	Rings, fields, and polynomials		207
	22.1 Rings		296
	22.2 Invertible elements of a ring		297
	22.3 Fields		299
	22.4 Polynomials		301 304
	22.5 The division algorithm for polynomials		304 306
	22.6 The Euclidean algorithm for polynomials		309
	22.7 Factorization of polynomials in theory		310
	22.8 Factorization of polynomials in practice		313
	22.9 Miscellaneous Exercises		313
23.	Finite fields and some applications		314
	23.1 A field with nine elements		
	23.2 The order of a finite field		315 317
	23.3 Construction of finite fields		317
	23.4 The primitive element theorem		322
	23.5 Finite fields and latin squares		344

XIV	Con	itents	
	23.6	Finite geometry and designs	324
		Projective planes	327
		Squares in finite fields	330
		Existence of finite fields	333
	23.10	Miscellaneous Exercises	336
24.	Erro	r-correcting codes	
		Words, codes, and errors	338
		Linear codes	341
		Construction of linear codes	343
		Correcting errors in linear codes	345
		Cyclic codes	349
		Classification and properties of cyclic codes	351
		Miscellaneous Exercises	355
25.	Gene	erating functions	
		Power series and their algebraic properties	357
		Partial fractions	360
		The binomial theorem for negative exponents	364
		Generating functions	367
		The homogeneous linear recursion	369
		Non-homogeneous linear recursions	372
	25.7	Miscellaneous Exercises	374
26.	Parti	tions of a positive integer	
		Partitions and diagrams	376
		Conjugate partitions	378
		Partitions and generating functions	379
		Generating functions for restricted partitions	382
		A mysterious identity	384
		The calculation of $p(n)$	387
	26.7	Miscellaneous Exercises	388
27.	Symi	metry and counting	
	27.1	The cycle index of a group of permutations	390
	27.2	Cyclic and dihedral symmetry	392
	27.3	Symmetry in three dimensions	395
	27.4	The number of inequivalent colourings	397
	27.5	Sets of colourings and their generating functions	400
	27.6	Pólya's theorem	402
	27.7	Miscellaneous Exercises	405
	Answ	vers to exercises	407
	Index	<b>(</b>	421