

Contents

Part One Groups, Rings, and Modules

CHAPTER I

Groups

| | |
|--|----|
| 1. Monoids | 5 |
| 2. Groups | 9 |
| 3. Cyclic groups | 13 |
| 4. Normal subgroups | 14 |
| 5. Operation of a group on a set | 19 |
| 6. Sylow subgroups | 23 |
| 7. Categories and functors | 25 |
| 8. Free groups | 33 |
| 9. Direct sums and free abelian groups | 40 |
| 10. Finitely generated abelian groups | 45 |
| 11. The dual group | 50 |

CHAPTER II

Rings

| | |
|--------------------------------------|----|
| 1. Rings and homomorphisms | 56 |
| 2. Commutative rings | 62 |
| 3. Localization | 66 |
| 4. Principal rings | 70 |

CHAPTER III

Modules

| | |
|--|----|
| 1. Basic definitions | 74 |
| 2. The group of homomorphisms | 76 |
| 3. Direct products and sums of modules | 79 |
| 4. Free modules | 84 |
| 5. Vector spaces | 85 |
| 6. The dual space | 88 |

CHAPTER IV

Homology

| | |
|--|-----|
| 1. Complexes | 94 |
| 2. Homology sequence | 95 |
| 3. Euler characteristic | 98 |
| 4. The Jordan-Hölder theorem | 102 |

CHAPTER V

Polynomials

| | |
|--|-----|
| 1. Free algebras | 106 |
| 2. Definition of polynomials | 110 |
| 3. Elementary properties of polynomials | 115 |
| 4. The Euclidean algorithm | 120 |
| 5. Partial fractions | 123 |
| 6. Unique factorization in several variables | 126 |
| 7. Criteria for irreducibility | 128 |
| 8. The derivative and multiple roots | 130 |
| 9. Symmetric polynomials | 132 |
| 10. The resultant | 135 |

CHAPTER VI

Noetherian Rings and Modules

| | |
|------------------------------------|-----|
| 1. Basic criteria | 142 |
| 2. Hilbert's theorem | 144 |
| 3. Power series | 146 |
| 4. Associated primes | 148 |
| 5. Primary decomposition | 152 |

Part Two

Field Theory

CHAPTER VII

Algebraic Extensions

| | |
|---|-----|
| 1. Finite and algebraic extensions | 161 |
| 2. Algebraic closure | 166 |
| 3. Splitting fields and normal extensions | 173 |
| 4. Separable extensions | 176 |
| 5. Finite fields | 182 |
| 6. Primitive elements | 185 |
| 7. Purely inseparable extensions | 186 |

CHAPTER VIII

Galois Theory

| | |
|---|-----|
| 1. Galois extensions | 192 |
| 2. Examples and applications | 199 |
| 3. Roots of unity | 203 |
| 4. Linear independence of characters | 208 |
| 5. The norm and trace | 210 |
| 6. Cyclic extensions | 213 |
| 7. Solvable and radical extensions | 216 |
| 8. Kummer theory | 218 |
| 9. The equation $X^n - a = 0$ | 221 |
| 10. Galois cohomology | 224 |
| 11. Algebraic independence of homomorphisms | 225 |
| 12. The normal basis theorem | 229 |

CHAPTER IX

Extensions of Rings

| | |
|---|-----|
| 1. Integral ring extensions | 237 |
| 2. Integral Galois extensions | 244 |
| 3. Extension of homomorphisms | 249 |

CHAPTER X

Transcendental Extensions

| | |
|--|-----|
| 1. Transcendence bases | 253 |
| 2. Hilbert's Nullstellensatz | 255 |
| 3. Algebraic sets | 257 |
| 4. Noether normalization theorem | 260 |
| 5. Linearly disjoint extensions | 261 |
| 6. Separable extensions | 264 |
| 7. Derivations | 266 |

CHAPTER XI

Real Fields

| | |
|---|-----|
| 1. Ordered fields | 271 |
| 2. Real fields | 273 |
| 3. Real zeros and homomorphisms | 278 |

CHAPTER XII

Absolute Values

| | |
|---|-----|
| 1. Definition, dependence, and independence | 283 |
| 2. Completions | 286 |

| | |
|--|-----|
| 3. Finite extensions | 292 |
| 4. Valuations | 296 |
| 5. Completions and valuations | 304 |
| 6. Discrete valuations | 305 |
| 7. Zeros of polynomials over complete fields | 308 |

Part Three

Linear Algebra and Representations

CHAPTER XIII

Matrices and Linear Maps

| | |
|--|-----|
| 1. Matrices | 321 |
| 2. The rank of a matrix | 323 |
| 3. Matrices and linear maps | 324 |
| 4. Determinants | 328 |
| 5. Duality | 337 |
| 6. Matrices and bilinear forms | 342 |
| 7. Sesquilinear duality | 346 |

CHAPTER XIV

Structure of Bilinear Forms

| | |
|---|-----|
| 1. Preliminaries, orthogonal sums | 354 |
| 2. Quadratic maps | 357 |
| 3. Symmetric forms, orthogonal bases | 358 |
| 4. Hyperbolic spaces | 359 |
| 5. Witt's theorem | 360 |
| 6. The Witt group | 363 |
| 7. Symmetric forms over ordered fields | 365 |
| 8. The Clifford algebra | 367 |
| 9. Alternating forms | 370 |
| 10. The Pfaffian | 372 |
| 11. Hermitian forms | 374 |
| 12. The spectral theorem (hermitian case) | 376 |
| 13. The spectral theorem (symmetric case) | 378 |

CHAPTER XV

Representation of One Endomorphism

| | |
|--|-----|
| 1. Representations | 384 |
| 2. Modules over principal rings | 386 |
| 3. Decomposition over one endomorphism | 395 |
| 4. The characteristic polynomial | 399 |

CHAPTER XVI

Multilinear Products

| | |
|---|-----|
| 1. Tensor product | 408 |
| 2. Basic properties | 412 |
| 3. Extension of the base | 418 |
| 4. Tensor product of algebras | 420 |
| 5. The tensor algebra of a module | 421 |
| 6. Alternating products | 424 |
| 7. Symmetric products | 427 |
| 8. The Euler-Grothendieck ring | 429 |
| 9. Some functorial isomorphisms | 431 |

CHAPTER XVII

Semisimplicity

| | |
|--|-----|
| 1. Matrices and linear maps over non-commutative rings | 438 |
| 2. Conditions defining semisimplicity | 441 |
| 3. The density theorem | 443 |
| 4. Semisimple rings | 446 |
| 5. Simple rings | 448 |
| 6. Balanced modules | 451 |

CHAPTER XVIII

Representations of Finite Groups

| | |
|--|-----|
| 1. Semisimplicity of the group algebra | 453 |
| 2. Characters | 455 |
| 3. One-dimensional representations | 459 |
| 4. The space of class functions | 461 |
| 5. Orthogonality relations | 465 |
| 6. Induced characters | 468 |
| 7. Induced representations | 471 |
| 8. Positive decomposition of the regular character | 475 |
| 9. Supersolvable groups | 478 |
| 10. Brauer's theorem | 480 |
| 11. Field of definition of a representation | 485 |

| | |
|---|-----|
| Appendix. The transcendence of e and π | 493 |
|---|-----|

| | |
|------------------------|-----|
| Index | 501 |
|------------------------|-----|