

Contents

Preface	ix
List of Grammatical Abbreviations	xi
Chapter 1. The Natural Numbers	1
1. Peano's axioms	1
2. Addition	2
3. Multiplication	4
4. Inequalities	6
5. Well Ordering	8
6. Some Notation from Set Theory	9
7. Counting	13
8. Finite Sums and Products	17
— Assignment 1	18
Supplementary Exercises	19
Chapter 2. Integers and Polynomials	20
1. Equivalence Relations	20
2. Definition of \mathbf{Z}	22
3. The Embedding $\mathbb{N} \rightarrow \mathbf{Z}$	25
4. The Archimedean Principle	26
— Assignment 2	29
5. Rings	31
6. Polynomials	35
— Assignment 3	41
Supplementary Exercises	44
Chapter 3. Rational Numbers and Rational Functions	46
1. Localization	47
2. Homomorphisms	49
3. Integral Domains and Fields	51

— Assignment 4	54
4. Quotient Rings	56
5. Ideals and Congruences	58
6. Groups and Units	63
7. Rational Arithmetic	70
8. Prime Ideals and Zorn's Lemma	71
— Assignment 5	74
9. Principal Ideal Domains	76
10. The Substitution Homomorphism and the Minimum Polynomial	79
11. Unique Factorization	84
12. Consequences of Unique Factorization	89
13. The Euclidean Algorithm	95
— Assignment 6	101
14. Polynomials over U.F.D.s	103
Supplementary Exercises	107

Chapter 4. Vector Spaces **113**

1. Systems of Linear Equations	113
2. Vector Spaces	118
3. Linear Dependence	122
4. Bases	124
5. Examples of Bases; Partial Fractions	126
6. Dimension	130
— Assignment 7	132
7. Partially Ordered Sets and Zorn's Lemma	135
8. The Schroeder–Bernstein Theorem	137
9. Infinite Dimension	141
10. Linear Transformations	142
11. Nullity and Rank	148
12. Direct Sums	152
— Assignment 8	155
13. Matrices	157
14. Elementary Matrices	165
15. Row Equivalence and Echelon Form	168
16. Applications of Echelon Form	174
17. Determinants	184
18. The Adjugate and Cramer's Rule	192
19. Permutations	195
— Assignment 9	204
20. Algebras	209
21. The Resultant	211
Appendix: The Equivalence of Zorn's Lemma and the Axiom of Choice	213
Supplementary Exercises	218

Chapter 5. The Real, Complex, and p -adic Numbers **226**

1. Introduction	226
2. Absolute Values	227

3. Completion	234
4. The Real Field	238
5. Decimal Expansions and g.l.b.s	243
6. Completeness	247
— Assignment 10	250
7. p -adic Numbers and Formal Power Series	252
8. Discretely Valued Fields	260
9. Absolute Convergence	267
10. Zeros of Polynomials; Rational Exponents	270
11. Newton's Method and Hensel's Lemma	286
12. Symmetric Polynomials	291
13. Root Fields; the Discriminant	296
14. The Complex Field	304
— Assignment 11	313
15. The Extended Binomial Theorem in Valued Fields	317
16. Metric Spaces	333
17. Complete Metric Spaces	336
18. Uniform Convergence	342
19. \log and \exp in Valued Fields	349
20. α^β in \mathbb{C}	357
21. Cyclotomy	367
22. Local Fields and the Bolzano–Weierstrass Theorem	382
23. Normed Vector Spaces and the Gelfand–Mazur Theorem	390
24. Further Group Theory and Wedderburn's Theorem	395
— Assignment 12	405
25. Newton's Polygon and Puiseux Expansions; Liouville Numbers	411
Supplementary Exercises	424
Table of Axiom Schemes	435
List of Symbols	437
Index	443