

Contents

Foreword	1
Advice to the reader	4
PART I. Foundations of Algebra	6
Further reading	7
Chapter 1. Sources of algebra	8
§1. Algebra in brief	9
§2. Some model problems	13
1. Solvability of equations in radicals	13
2. The states of a molecule	15
3. Coding information	16
4. The heated plate problem	17
§3. Systems of linear equations. The first steps	17
1. Terminology	18
2. Equivalence of linear systems	20
3. Reducing to step form	22
4. Studying a system of linear equations	24
5. Some remarks and examples	27
§4. Determinants of small order	29
Exercises	34
§5. Sets and mappings	35
1. Sets	35
2. Mappings	37
Exercises	42
§6. Equivalence relations. Quotient maps	43
1. Binary relations	44
2. Equivalence relations	44
3. Quotient maps	46
4. Ordered sets	48
Exercises	49
§7. The principle of mathematical induction	50
§8. Integer arithmetic	54
1. The fundamental theorem of arithmetic	55
2. g.c.d. and l.c.m. in \mathbb{Z}	56
3. The division algorithm in \mathbb{Z}	57
Exercises	58
Chapter 2. Vector spaces. Matrices	60
§1. Vector spaces	61
1. Motivation	61
2. Basic definitions	61
3. Linear combinations. Linear span	64
4. Linear dependence	66
5. Bases. Dimension	67
Exercises	70
§2. The rank of a matrix	71
1. Back to equations	71
2. The rank of a matrix	73
3. Solvability criterion	76
Exercises	77

§3. Linear maps. Matrix operations	78
1. Matrices and maps	78
2. Matrix multiplication	81
3. Square matrices	84
Exercises	91
§4. The space of solutions	93
1. Solving a homogeneous linear system	93
2. Linear manifolds. Solving a non-homogeneous system	97
3. The rank of a product of matrices	98
4. Equivalence classes of matrices	99
Exercises	104
 Chapter 3. Determinants	106
§1. Determinants: construction and basic properties ..	107
1. Construction by induction	107
2. Basic properties of determinants	110
Exercises	118
§2. Further properties of determinants	118
1. Expanding the determinant along an arbitrary column	118
2. The properties of determinants relating to columns	119
3. The transpose determinant	120
4. Determinants of special matrices	124
5. Building up a theory of determinants	128
Exercises	129
§3. Applications of determinants	130
1. Criterion for a matrix to be non-singular	130
2. Computing the rank of a matrix	134
Exercises	136
 Chapter 4. Algebraic structures (groups, rings, fields) ...	138
§1. Sets with algebraic operations	138
1. Binary operations	138
2. Semigroups and monoids	139
3. Generalized associativity; powers	141
4. Invertible elements	144
Exercises	144
§2. Groups	145
1. Definition and examples	145
2. Systems of generators	148
3. Cyclic groups	150
4. The symmetric group and the alternating group ..	153
Exercises	162
§3. Morphisms of groups	164
1. Isomorphisms	164
2. Homomorphisms	169
3. Glossary. Examples	170
4. Cosets of a subgroup	173
5. The monomorphism $S_n \rightarrow \text{GN}(n)$	177
Exercises	181
§4. Rings and fields	182
1. The definition and general properties of rings	182
2. Congruences. The ring of residue classes	187
3. Ring homomorphisms and ideals	189
4. The concept of quotient group and quotient ring	191
5. Types of rings. Fields	195

6. The characteristic of a field	199
7. A remark on linear systems	202
Exercises	205
 Chapter 5. Complex numbers and polynomials	207
§1. The field of complex numbers	207
1. An auxiliary construction	208
2. The complex plane	209
3. Geometrical interpretation of operations with complex numbers	210
4. Raising to powers and extracting roots	214
5. Uniqueness theorem	217
Exercises	220
§2. Rings of polynomials	222
1. Polynomials in one variable	223
2. Polynomials in several variables	228
3. The division algorithm	232
Exercises	235
§3. Factoring in polynomial rings	237
1. Elementary divisibility properties	237
2. g.c.d. and l.c.m. in rings	241
3. Unique factorization in Euclidean rings	243
4. Irreducible polynomials	247
Exercises	251
§4. The field of fractions	252
1. Construction of the field of fractions of an integral domain	252
2. The field of rational functions	255
3. Primary rational functions	257
Exercises	261
 Chapter 6. Roots of polynomials	263
§1. General properties of roots	263
1. Roots and linear factors	263
2. Polynomial functions	266
3. Differentiation in polynomial rings	269
4. Multiple factors	271
5. Vieta's formulas	274
Exercises	277
§2. Symmetric polynomials	279
1. The ring of symmetric polynomials	279
2. The fundamental theorem on symmetric polynomials	281
3. The method of undetermined coefficients	284
4. The discriminant of a polynomial	288
5. The resultant	291
Exercises	295
§3. \mathbb{C} is algebraically closed	296
1. Statement of the fundamental theorem	296
2. The splitting field of a polynomial	299
3. Proof of the Fundamental Theorem	303
§4. Polynomials with real coefficients	307
1. Factorization in $\mathbb{R}[X]$	307
2. The problem of isolating the roots of a polynomial	309
3. Stable polynomials	315
Exercises	317

PART II. Groups, Rings, Modules	320
Further reading	321
Chapter 7. Groups	322
§1. Classical groups in low dimensions	322
1. General definitions	322
2. Parametrization of $SU(2)$ and $SO(3)$	324
3. The epimorphism $SU(2) \rightarrow SO(3)$	326
4. Geometrical characterization of $SO(3)$	329
Exercises	330
§2. Group actions on sets	331
1. Homomorphisms $G \rightarrow S(\Omega)$	331
2. The orbit and stationary subgroup of a point ..	332
3. Examples of group actions on sets	335
4. Homogeneous spaces	340
Exercises	341
§3. Some group theoretic constructions	342
1. General theorems on group homomorphisms	343
2. Solvable groups	348
3. Simple groups	351
4. Products of groups	354
5. Generators and defining relations	357
Exercises	364
§4. The Sylow theorems	368
Exercises	375
§5. Finite abelian groups	376
1. Primary abelian groups	376
2. The structure theorem for finite abelian groups	381
Exercises	384
Chapter 8. Elements of representation theory	386
§1. Definitions and examples of linear representations	390
1. Basic concepts	390
2. Examples of linear representations	396
Exercises	403
§2. Unitary and reducible representations	404
1. Unitary representations	404
2. Complete reducibility	408
Exercises	412
§3. Finite rotation groups	412
1. The orders of finite subgroups of $SO(3)$	413
2. Symmetry groups for regular polyhedra	416
Exercises	420
§4. Characters of linear representations	421
1. Schur's lemma and corollary	421
2. Characters of representations	424
Exercises	432
§5. Irreducible representations of finite groups	433
1. The number of irreducible representations	433
2. The degrees of the irreducible representations	435
3. Representations of abelian groups	438
4. Representations of certain special groups	441
Exercises	445
§6. Representations of $SU(2)$ and $SO(3)$	448
Exercises	453

§7. Tensor products of representations	453
1. The dual representation	453
2. Tensor products of representations	454
3. The ring of characters	459
4. Invariants of linear groups	463
Exercises	469
 Chapter 9. Toward a theory of fields, rings and modules ...	472
§1. Finite field extensions	472
1. Primitive elements and the degree of an extension	472
2. Isomorphism of splitting fields	477
3. Finite fields	480
4. The Möbius inversion formula and its applications	485
Exercises	493
§2. Various results about rings	496
1. More examples of unique factorization domains ..	496
2. Ring theoretic constructions	501
3. Number theoretic applications	504
Exercises	509
§3. Modules	512
1. Basic facts about modules	512
2. Free modules	517
3. Integral elements of a ring	521
4. Unimodular sequences of polynomials	522
§4. Algebras over a field	524
1. Definitions and examples of algebras	524
2. Division rings (skew fields)	527
3. Group algebras and modules over them	531
4. Non-associative algebras	539
Exercises	545
 Appendix. The Jordan normal form of a matrix	548
Hints to the exercises	561
Index	573